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Proceedings of the American Academy of Arts and Sciences

Vol. 52. No. 6. — November, 1916.

DIFFERENTIAL GEOMETRY OF TWO DIMENSIONAL SURFACES IN HYPERSPACE.

By Edwin B. Wilson and C. L. E. Moore.

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VOLUME 52.

DIFFERENTIAL GEOMETRY OF TWO DIMENSIONAL SURFACES IN HYPERSPACE.

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Received March 25, 1916.

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1. Introduction. There are several ways of generalizing the ordinary differential theory of surfaces. The one most extensively treated is that which deals with varieties of $n-1$ dimensions in a Euclidean space of n dimensions.[1] A second method is to investigate properties of two-dimensional varieties in a space of four or indeed of n dimensions.[2] A third and more general extension of the theory would be to study varieties of k dimensions in a space of n dimensions, and under this head a very interesting species can arise [3] when $n = 2k-1$. The recent contributions to this third have dealt with the projective differential properties and thus have afforded only a partial generalization of the general theory.

We propose here to study the theory of two-dimensional varieties in space of n dimensions and to exhibit the way in which the ordinary theory arises through specialization. The generalization in this case is not so immediately obvious as in the first case and perhaps throws more light on the ordinary theory of surfaces than does that.

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1 See, for instance, Killing, *Die Nicht-Euklidischen Raumformen;* Bianchi, *Lezioni di Geometria Differenziale,* Vol. I, Chaps. 11, 14; Shaw, *Amer. J. Math.,* **35**, No. 4, 395–406.

2 K. Kommerell, Die Krümmung der Zweidimensionalen Gebilde, in ebenen Raum von vier Dimensionen, Dissertation, Tübingen, 1897, 53 pp.; and Riemannsche Flächen in ebenen Raum von vier Dimensionen, *Math. Ann.,* **60**, in which the dissertation is also summarized; E. E. Levi, Saggio sulla Teoria delle Superficie a due Dimensioni immersi in un Iperspazio, *Ann. R. Scu. Norm.,* Pisa, **10**, 99 pp.; C. L. E. Moore, *Ann. Math.* (2) **16**, 89–96 (1915).

3 C. Segre, Su una Classe di Superficie ecc., *Att. Torino,* **42** (1907), and *Rend. Circ. Mat.,* **30**, 87–121 (1910); and further developments by Bompiani and Terracini.

2. Methods of attack. When attacking the theory of the two-surface in S_n, the method of attack is of fundamental importance. That followed by Kommerell consists in starting with the finite equations of the surface and in trying by geometric intuition to find what sort of properties lend themselves most readily to generalization. This method has the disadvantage that it is somewhat lacking in system, and one is never confident that he is not overlooking things that are perhaps the most vital to the subject. Levi starts with the finite parametric equations of the surface and determines the invariants I of orthogonal transformations and the elements J covariant under a change of parameters. This is more systematic and safer.

It seems clear that the safest and most systematic method of attack is to discuss the surface entirely from the point of view of the differential quadratic form or better of the set of differential quadratic forms which define the surface. In following this method we have the advantage that Ricci, in his *Lezioni sulla Teoria delle Superficie,*[4] has pursued more consistently than any one else the same method with regard to surfaces in ordinary space. In his work those properties which depend on the first fundamental form are first developed, and then those which follow from the first and second forms together. Now the first fundamental form defines a surface in so far and only so far as that surface may one of the infinite class of surfaces applicable upon it. Thus the first fundamental form determines a surface as a

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4 Padova, Drucker, 1898 (Lithographed). The contents of this book is as follows:— *Introduzione:* I. Delle equazioni lineari ed omogenee a derivate parziali di I. ordine e dei sistemi completi, p. 1. II. Nozioni generali sulle forme differenziali quadratiche, p. 36. III. Del calcolo differenziale assoluto ad n variabili, p. 45. IV. Della classificazione delle forme differenziali quadratiche positive, p. 73. V. Degli invarianti assoluti comuni ad una forma fondamentale ed ai sistemi associati, p. 91. VI. Del calcolo differenziali assoluto a due variabili independenti, p. 105. *Parte Prima: Della proprietà delle superficie considerate come veli flessibili ed inestendibili.* I. Dei sistemi di coordinate sopra una superficie qualunque, p. 134. II. Generalità sulle congruenze di linee tracciate sopra una superficie, p. 148. III. Considerazioni generali sugli invarianti differenziali ecc., p. 163. IV. Delle congruenze di linee geodetiche e di linee parallele, p. 176. V. Fascii e sistemi isotermi, e rappresentazioni conformi, p. 202. VI. Sulla integrazione della equazione delle congruenze geodetiche, p. 223. VII. Delle congruenze isoterme di Liouville, p. 248. *Parte Seconda: Teoria delle superficie considerate come dotate di forma rigida nello spazio.* I. Equazioni generali della teoria delle superficie, p. 270. II. Delle linee di curvatura e delle linee asintotiche, p. 287. III. Della rappresentazione sferica di Gauss, p. 309. IV. Di alcune classi speciali di superficie, p. 322. V. Evolute e superficie di Weingarten, p. 350. VI. Delle superficie di secondo grado, p. 366. VII. Della applicabilità delle superficie, p. 385.

perfectly flexible inextensible membrane. There is no restriction to a rigid surface in space and none upon the number of dimensions in which the surface may lie; we work entirely on the surface itself. Hence all the results which Ricci obtained in Part I of his *Lezioni* are true without any modification of the proofs or the interpretation in any number of dimensions.

3. Quadratic differential forms. When we wish to interpret a manifold defined by a binary quadratic differential form as a surface in space we have to introduce a set of variables such that

$$ds^2 = \Sigma_{ij}\, a_{ij} dx_i dx_j = dy_1^2 + dy_2^2 + dy_3^2, \qquad i,j = 1, 2;$$

and it is the determination of this set of variables which leads to the second fundamental form. It is a fundamental proposition in the theory of binary quadratic forms that such a form may be written as the sum of three squares. Hence for the interpretation of a binary differential form as a surface, three dimensions are sufficient. When the theory of the ternary differential quadratic form is studied with reference to its reduction to a sum of squares, it is found that in general *six* variables are needed. Hence to interpret the theory of the ternary form we must in general go to a spread V_3 of three dimensions in S_6. It is clear from this that the theory of the V_3 in S_4 does not correspond with the theory of any but a very special class of ternary forms. Hence from the point of view of the quadratic form the theory of surfaces does not generalize very simply. In general for a quadratic differential form in k variables the reduction to the sum of squares may require $k(k-1)/2$ variables, and the minimum number of additional variables required, above k, is called the *class* of the form.[5]

When in our special case of a binary quadratic form, we wish to interpret the form as a surface in S_n, we have to determine n variables y so that $ds^2 = \Sigma_i\, dy_i^2$, $i = 1, \cdots, n$. The determination is made by the properties of systems of partial differential equations, in particular complete systems. This has been accomplished by Ricci in the general case and his result is stated in a theorem.[6]

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5 Ricci, *Lezioni*, Introduction, Chap. 4.

6 Ricci, *Lezioni*, pp. 90–91. Ricci has also treated the more general question of a variety of n dimensions immersed in a variety (not a Euclidean space) of $n + m$ dimensions and the transformation of $\Sigma a_{rs} dx_r dx_s$, $r, s = 1, \ldots, n$, into $\Sigma c_{uv} dy_u dy_v$, $u, v = 1, \ldots, n + m$. *Rend. R. Acc. Lincei*, (5) **11**, 355–362 (1902).

We shall not make a direct use of that theorem but shall give an independent demonstration of the special case in which we are interested. For in order properly to understand the statement of this theorem it would be necessary to explain the technical language of Ricci's absolute differential calculus, and we consider it better to explain this piece by piece as we need it, and to give at length demonstrations of theorems which are special cases of his, in order that we may make the work less abstract.

CHAPTER I. RICCI'S METHOD.[7]

4. Two types of transformations. If by a change of variable,

$$x_1 = x_1(y_1, y_2), \quad x_2 = x_2(y_1, y_2),$$

we transform the differential $X_1 dx_1 + X_2 dx_2$ into a new differential in the new variables so that

$$X_1 dx_1 + X_2 dx_2 = Y_1 dy_1 + Y_2 dy_2,$$

we find

$$Y_1 = X_1 \frac{\partial x_1}{\partial y_1} + X_2 \frac{\partial x_2}{\partial y_1}, \quad Y_2 = X_1 \frac{\partial x_1}{\partial y_2} + X_2 \frac{\partial x_2}{\partial y_2}; \tag{1}$$

and if by the same change of variable we transform the differential system

$$\frac{dx_1}{X^{(1)}} = \frac{dx_2}{X^{(2)}} \quad \text{into} \quad \frac{dy_1}{Y^{(1)}} = \frac{dy_2}{Y^{(2)}},$$

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7 The lithographed *Lezioni* already cited is not obtainable either in new or second hand copies and is to be found in very few American libraries; it is to be had, however, at the Harvard library, the Boston Public library, and the library of Washington University (St. Louis). Ricci's first presentation of the essentials of the theory is scattered through a considerable number of papers in different Italian journals, particularly journals of the learned societies. See, e. g., *Rend. R. Acc. Lincei*, **5**, 112–118 (1889); *Studi off. d. Univ. Padovana a. Bolognese n. VIII centenario ecc.*, Vol. III (1888); Atti R. Ist. Veneto, (7) **4**, 1–29 (1897), *Ibid.*, **5**, 643–681 (1894), *Ibid.*, **6**, 445–488 (1895); *Rend. R. Acc. Lincei* (5) **4**, 232–237 (1895), *Ibid.*, **11**, 355–362 (1902). A general sketch of the method is found in *Bull. Sci. Math., Paris,* (2) **16**, 167–189 (1892) and a very elaborate outline not only of the foundations of the theory but of many of its applications is given by Ricci and Levi-Civita in *Math. Ann.* **54**, 125–201 (1900). More recently Grossmann, *Verallgem. Relativitätstheorie* (with Einstein), Teubner, 1913 (from *Zs. Math. Physik*, Vol. 62), mentions a few of the salient features of the method in a modified notation. It is however only in the *Lezioni* that the treatment of the elementary parts of the theory is given in comfortable detail. Moreover, the *Math. Encyc.* and the few authors who cite Ricci do so in a manner which suggests strongly that his method is practically unknown. These facts are offered in justification of our reproducing here material which has previously been published.

we find, on working out the details,

$$Y^{(1)} = X^{(1)}\frac{\partial y_1}{\partial x_1} + X^{(2)}\frac{\partial y_1}{\partial x_2}, \quad Y^{(2)} = X^{(1)}\frac{\partial y_2}{\partial x_1} + X^{(2)}\frac{\partial y_2}{\partial x_2}. \tag{2}$$

We can write the transformation from X to Y in the two cases in the forms

$$Y_r = \Sigma_s X_s \frac{\partial x_s}{\partial y_r}, \tag{1'}$$

$$Y^{(r)} = \Sigma_s X^{(s)} \frac{\partial y_r}{\partial x_s}. \tag{2'}$$

The system X_1, X_2 is said to be transformed *covariantly*. The system $Y^{(1)}$, $Y^{(2)}$ is said to be transformed *contravariantly*. Furthermore if we have any system of X's which is so defined that the transformed system follows the rule (1′), it is called *covariant* and the members of the system are denoted by lower indices; whereas if the system follows the rule of transformation (2′), it is called *contravariant* and the members of the system are denoted by upper indices. The system of differentials dx_1, dx_2, by the formula for the total differential, is seen to follow rule (2′) and therefore the system of differentials of the independent variables are the members of a contravariant system; but we observe that the indices are lower, in conformity with ordinary practice, and not upper as the rule here would require.

If we were dealing with more than two variables x_1, x_2, we should still find that the transformation of the differential

$$X_1dx_1 + X_2dx_2 + \cdot\cdot\cdot\cdot\cdot + X_ndx_n$$

led to the rule (1′), where s runs from 1 to n, for changing X's into Y's; and that the transformation of the system of equations

$$\frac{dx_1}{X^{(1)}} = \frac{dx_2}{X^{(2)}} = \cdot\cdot\cdot\cdot\cdot = \frac{dx_n}{X^{(n)}}$$

led to the set of equations (2′), where s runs from 1 to n.

5. Generalization of vector analysis. If we could follow the ideas of Grassmann-Gibbs,[8] we should consider the sets of quantities

$$X_1, X_2, \cdot \cdot \cdot, X_n \quad \text{or } X^{(1)}, X^{(2)}, \cdot \cdot \cdot, X^{(n)}$$

as components of a vector along the directions dx_i or upon the planes perpendicular to those directions. It proves, however, to be impossible to establish here more than an analogy; for it is actually untrue that these elements are such components.

That the X's may not generally be interpreted as components of a vector is clear from the expression for the differential of work in terms of generalized coördinates, namely,

$$dW = Q_1 dq_1 + Q_2 dq_2 + \ldots + Q_n dq_n.$$

The set of generalized forces Q_i is covariant under a transformation of the q's, but the generalized forces are not the projections of the resultant force either upon the directions dq_i or upon the planes perpendicular thereto in the n-dimensional representative space of the q's; for instance, in polar coördinates in the plane, $dW = R\, dr + r\Theta d\theta$, where R and Θ (not $r\Theta$) are the radial and tangential components of the force.

We have therefore to deal not with a generalized vector-analysis but with a generalization of vector analysis when we deal with systems X_s or $X^{(s)}$. A method of converting such a system into one which represents the components of a vector will be mentioned later (note 17 to § 12).

So long as we remain in the vicinity of a particular point and deal only with differentials of the first order, the transformations (1′) and (2′) are linear with constant coefficients of the type

$$Y_r = \Sigma_s\, c_{sr} X_s, \qquad c_{sr} = \frac{\partial x_s}{\partial y_r},$$

$$Y^{(r)} = \Sigma_s\, \gamma_{sr} X^{(s)}, \qquad \gamma_{sr} = \frac{\partial y_r}{\partial x_s},$$

and the first section of our presentation of Ricci's method will therefore be strictly algebraic theory of the linear transformation. When,

[8] Grassmann, *Ausdehnungslehren von* 1844 *u.* 1862, also *Gesammelte Werke;* Gibbs, *Scientific Papers,* Vol. II; Gibbs-Wilson, *Vector Analysis;* Wilson, *Trans. Conn. Acad. Arts Sci.,* **14**, 1–57, (1908).

later, we come to differentiation we shall have to take into account the variability of the coefficients of the transformation.

6. Sets of elements. We deal with sets of elements of different orders; thus our system is a generalization of matrical as well as of vectorial analysis. The fundamental elements are sets of quantities X with m indices, each of which may take all values from 1 to n. For example, if $n = 2$,

$m = 0,$	$X,$	no index;
$m = 1,$	$X_1, \quad X_2,$	one index;
$m = 2,$	$X_{11}, X_{\cancel{21}\,12}, X_{21}, X_{22},$	two indices;
$m = 3,$	$X_{111}, X_{112}, X_{121}, X_{122}, X_{211}, X_{212}, X_{221}, X_{222},$	three indices.

In general there are n^m quantities in the system of order m with 1 to n as the range for each index.

These systems of successive orders are analogous to the scalars, vectors, dyadics, triadics, ... of Gibbs, and to the scalars, vectors, open products with 1, 2, . . . openings of Grassmann; the matrical analogy would take us to matrices of higher dimensions than the usual two. The *addition* of two systems of the same order and the multiplication of a system by a constant are according to definitions obviously suggested by the analogies, i. e., the systems are *linear*.

Multiplication [9] of a system of order m into a system of order m' consists in multiplying each element of the first system into each element of the second and gives a system of order $m + m'$. For example,

$$(X_1, X_2)\,(X_{11}, X_{12}, X_{22}, X_{22}) = X_1X_{11}, X_1X_{12}, X_1X_{21}, X_1X_{22}, X_2X_{11}, X_2X_{12}, X_2X_{21}, X_2X_{22},$$

which is a system of order 3 and may be written $X_{ijk}, \quad i, j, k = 1, 2.$

By following the method of Gibbs [10] we may construct an outer or "combinatory" product of two systems of order 1, as

$$(X_1, X_2, X_3) \times (Y_1, Y_2, Y_3) = X_2Y_3 - X_3Y_2, \quad X_3Y_1 - X_1Y_3, \quad X_1Y_2 - X_2Y_1$$

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9 Grossmann calls the multiplication "outer" from analogy with Grassmann's outer product with which it has little in common; the real analogy is with Gibbs' indeterminate and Grassmann's open product.

10 On Multiple Algebra, *Scientific Papers*, Vol. II, pp. 91–117.

for the case $n = 3$; and for the general case the elements of the product would be $X_{ij} = X_i Y_j - X_j Y_i$. This system of the second order is skew symmetric, that is, $X_{ij} = -X_{ji}$, $X_{jj} = 0$. We could likewise form an "algebraic" product $X_{ij} = X_i Y_j + X_j Y_i$, which is symmetric. And in general we could form the combinatory and algebraic products of k systems.

If we wish actually to write the systems as hypercomplex numbers with "units" attached, we have

$$X_1 e_1 + X_2 e_2,$$
$$X_{11} e_1 e_1 + X_{12} e_1 e_2 + X_{21} e_2 e_1 + X_{22} e_2 e_2,$$

and so on. The product of these two systems would be similarly expressed with the units $e_1 e_1 e_1$, $e_1 e_1 e_2$, . . . exactly as the triadic which arises from the product of a vector and a dyadic.

If we wish to consider the units e_1, e_2, or $e_1 e_1$, $e_1 e_2$, $e_2 e_1$, $e_2 e_2$, etc., replaced by the set of independent variables, x_1, x_2, or $x_1 y_1$, $x_1 y_2$, $x_2 y_1$, $x_2 y_2$, etc., the expressions become

$$X_1 x_1 + X_2 x_2, \qquad X_{11} x_1 y_1 + X_{12} x_1 y_2 + X_{21} x_2 y_1 + X_{22} x_2 y_2,$$

and so on,— that is, they become linear, bilinear, trilinear, . . ., forms. Ricci's system of the mth order with range 1 to n is therefore analogous to an m-linear form in n variables.

7. Transformations of sets of elements. Consider next the linear transformation [11]

$$x_i = \Sigma_j c_{ij} y_j. \tag{3}$$

These equations may be solved by multiplying by the cofactors C_{ik} each divided by the determinant $|c_{ij}|$, that is, by $\gamma_{ik} = C_{ik}/|c_{ij}|$, and summing with respect to i. Then

$$y_k = \Sigma_i \gamma_{ik} x_i \quad \text{or } y_i = \Sigma_j \gamma_{ji} x_j. \tag{3'}$$

If u_i, v_i are variables contragredient to x_i and y_i, the transformation upon the u's and v's is

$$v_i = \Sigma_j c_{ji} u_j \quad \text{or } u_i = \Sigma_j \gamma_{ij} v_j. \tag{4}$$

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11 We may refer to Bôcher's *Introduction to Higher Algebra* for the theory of linear transformations, linear dependence, cogredient and contragredient variables, bilinear forms, square matrices, etc.

If the transformation (3) be effected upon the variables of the linear, bilinear, ..., m-linear forms, there arise new forms in which the coefficients are (X_i), (X_{ij}), ..., if we now use Ricci's notation,[12] in place of Y_i, Y_{ij} used above. The law of transformation between the X's and (X)'s is important and is obtained as follows:

$$\Sigma_i X_i x_i = \Sigma_i X_i \Sigma_j c_{ij} y_j = \Sigma_j (\Sigma_i c_{ij} X_i) y_j = \Sigma_j (X_j) y_j.$$

Hence

$$(X_j) = \Sigma_i c_{ij} X_i \text{ or } (X_i) = \Sigma_j c_{ji} X_j. \tag{5}$$

If we solve, we have

$$X_i = \Sigma_j \gamma_{ij} (X_j). \tag{5'}$$

Similarly if we take a bilinear form, we find

$$\begin{aligned}\Sigma_{ij} X_{ij} x_i \xi_j &= \Sigma_{ij} X_{ij} \Sigma_k c_{ik} y_k \Sigma c_{jl} \eta_l \\ &= \Sigma_{kl} (\Sigma_{ij} c_{ik} c_{jl} X_{ij}) y_k \eta_l = \Sigma_{kl} (X_{kl}) y_k \eta_l.\end{aligned}$$

Hence changing subscripts we have

$$(X_{ij}) = \Sigma_{kl} c_{ki} c_{lj} X_{kl}, \qquad X_{ij} = \Sigma_{kl} \gamma_{ik} \gamma_{jl} (X_{kl}). \tag{6}$$

In general for a system of order m, the transformation of the m-linear form shows that the transformation of the system follows the rule

$$(X_{i_1 i_2 \ldots i_m}) = \Sigma_{j_1 j_2 \ldots j_m} c_{j_1 i_1} c_{j_2 i_2} \ldots c_{j_m i_m} X_{j_1 j_2 \ldots j_m} \tag{7}$$

or

$$X_{i_1 i_2 i_3 \ldots i_m} = \Sigma_{j_1 j_2 \ldots j_m} \gamma_{i_1 j_2} \gamma_{i_2 j_2} \ldots \gamma_{i_m j_m} (X_{j_1 j_2 \ldots j_m}). \tag{7'}$$

The results of this article may be written more compactly in matrical notation. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, with a similar meaning for $\mathbf{y}$, be an extensive magnitude. Let $\mathbf{M}$ be the matrix

$$\mathbf{M} = \left\| \begin{matrix} c_{11} \; c_{12} \ldots c_{1n} \\ c_{21} \; c_{22} \ldots c_{2n} \\ \ldots\ldots\ldots\ldots \\ c_{n1} \; c_{n2} \ldots c_{nn} \end{matrix} \right\|$$

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12 Ricci, *Lezioni*, p. 49. Although the use of ( ) for the transformed quantities appears awkward it is less so than any notation which has occurred to us.

of the coefficients of (3). Let $\mathbf{M}^{-1}$ and $\mathbf{M}_C$ be the reciprocal and conjugate. Then,

$$\mathbf{x} = \mathbf{M}\cdot\mathbf{y}, \quad \mathbf{y} = \mathbf{M}^{-1}\cdot\mathbf{x}, \quad \mathbf{u} = \mathbf{v}\cdot\mathbf{M}^{-1}, \quad \mathbf{v} = \mathbf{u}\cdot\mathbf{M},$$

provided the products are defined as usual and the dot is used in the sense of Gibbs. The transformations of $\mathbf{X}$ and $\mathbf{XY}$ into $(\mathbf{X})$ and $(\mathbf{XY})$ are

$$\mathbf{X}\cdot\mathbf{x} = \mathbf{X}\cdot\mathbf{M}\cdot\mathbf{y} = (\mathbf{X})\cdot\mathbf{y}, \quad (\mathbf{X}) = \mathbf{X}\cdot\mathbf{M}, \quad \mathbf{X} = (\mathbf{X})\cdot\mathbf{M}^{-1},$$
$$\mathbf{XY}:\mathbf{x}\boldsymbol{\xi} = \mathbf{XY}:[\mathbf{M}\cdot\mathbf{y}\ \mathbf{M}\cdot\boldsymbol{\eta}] = (\mathbf{XY}):\mathbf{y}\boldsymbol{\eta},$$
$$(\mathbf{XY}) = \mathbf{XY}:\mathbf{MM}, \quad \mathbf{XY} = (\mathbf{XY}):\mathbf{M}^{-1}\mathbf{M}^{-1}.$$

The double products (containing two dots) are to be interpreted as indicating the union of *corresponding* elements, that is,

$$\mathbf{XY}:\mathbf{MM} = [\mathbf{X}\cdot\mathbf{M}]\,[\mathbf{Y}\cdot\mathbf{M}] \qquad \text{and} \qquad \mathbf{XY}:\mathbf{x}\boldsymbol{\xi} = [\mathbf{X}\cdot\mathbf{x}]\,[\mathbf{Y}\cdot\boldsymbol{\xi}].$$

The expression $\mathbf{XY}:\mathbf{MM}$ may be written also as $\mathbf{M}_C\cdot\mathbf{XY}\cdot\mathbf{M}$. The use of a formal product of the dyad type $\mathbf{XY}$ for any system of the second order is legitimate because the systems are linear.

8. **An adjoined quadratic form.** Now if we have a given fundamental quadratic form

$$\Sigma_{ij}a_{ij}x_ix_j, \qquad a_{ij} = a_{ji}, \tag{8}$$

the transformation of the coefficients a_{ij} will be the same as that of the elements X_{ij} found above. It is for this reason that we say that the system X_{ij} transforms *covariantly* with a_{ij} or with the given form (8); and we shall further say that a system of X's with any number of (lower) indices which transforms according to (7) or (7′) is a *covariant system.* The simplest case, given by (5) or (5′), shows that the covariant system of order 1 is transformed like the contragredient variables in (4).

If now we have a system, which we may denote by $X^{(i)}$, instead of by X_i, which transforms like the cogredient variables, or in an analogous manner, we call the system *contravariant*;[13] that is if,

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13 Grossmann employs Greek letters with lower indices to designate a contravariant system instead of Ricci's letters with upper indices. A trial of this notation convinces us that however awkward Ricci's notation appears it is more convenient than that of Grossmann; especially in view of the principal of duality (§ 9), the lack of correspondence between Greek and Roman letters, and the undesirability of immobilizing alphabets in a definite sense.

$$(X^{(i)}) = \Sigma_j \gamma_{ji} X^{(j)}, \qquad X^{(i)} = \Sigma_j c_{ij} (X^{(j)}), \tag{9}$$

$$(X^{(ij)}) = \Sigma_{kl} \gamma_{ki} \gamma_{lj} X^{(kl)}, \qquad X^{(ij)} = \Sigma_{kl} c_{ik} c_{jl} (X^{(kl)}), \tag{10}$$

and

$$(X^{(i_1 i_2 \,.\,.\, i_m)}) = \Sigma_{j_1 j_2 \,.\,.\, j_m} \gamma_{j_1 i_1} \gamma_{j_2 i_2} \,.\,.\,.\, \gamma_{j_m i_m} X^{(j_1 j_2 \,.\,.\, j_m)}, \tag{11}$$

$$X^{(i_1 i_2 \,.\,.\, i_m)} = \Sigma_{j_1 j_2 \,.\,.\,.\, j_m} c_{i_1 j_1} c_{i_2 j_2} \,.\,.\,.\, c_{i_m j_m} (X^{(j_1 j_2 \,.\,.\, j_m)}), \tag{11'}$$

these systems X with upper indices are contravariant of orders 1, 2, and m, respectively.

An important contravariant system is formed of the elements a_{ij} which are the cofactors of the determinant of the quadratic form (8) each divided by the determinant of the form. We may prove this as follows. Let ϵ_{ik} be 0 or 1 according as $j \neq k$ or $j = k$. Then,

$$\Sigma_i a_{ij} \alpha_{ik} = \epsilon_{jk}.$$

Substitute for a_{ij} from (6). Then

$$\Sigma_i \Sigma_{pq} \gamma_{ip} \gamma_{jq} (a_{pq}) \alpha_{ik} = \epsilon_{jk}.$$

Multiply by c_{js} and sum over j; the expression reduces by virtue of the fact that $\Sigma_j c_{js} \gamma_{jq}$ is zero unless $s = q$ and unity if $s = q$. (We have therefore

$$\Sigma_{jq} \gamma_{jq} c_{js} (a_{pq}) = (a_{ps}) \tag{12}$$

which is a formula often used for reducing certain double sums to a single term.) Hence

$$\Sigma_i \Sigma_p \gamma_{ip} (a_{ps}) \alpha_{ik} = \Sigma_j \epsilon_{jk} c_{js}.$$

Multiply by (α_{ts}) and sum over s; then $p = t$ alone gives something. (We have then

$$\Sigma_{sp} \gamma_{ip} (a_{ps}) (\alpha_{ts}) = \gamma_{it} \tag{13}$$

which like (12) reduces a double sum to a single term.) Hence

$$\Sigma_i \gamma_{it} \alpha_{ik} = \Sigma_{js} \epsilon_{jk} c_{js} (\alpha_{ts}).$$

Multiply by c_{rt} and sum over t. The double sum Σ_{it} on the left reduces to the single term a_{rk} by (12), and we have

$$a_{rk} = \Sigma_{jst}\epsilon_{jk}c_{js}c_{rt}(a_{ts}).$$

Now $j = k$ alone contributes something. Hence, finally,

$$a_{rk} = \Sigma_{st}c_{ks}c_{rt}(a_{ts}).$$

If we compare this with (10), we see that the transformation of the a's is contravariant, and the theorem is proved. We shall therefore in conformity with our general notation use upper indices and indeed write

$$a_{ij} = a^{(ij)}$$

for the cofactor of a_{ij} in (8) divided by the determinant $|a_{ij}|$.

If $\mathbf{X}^\circ = X^{(1)}, X^{(2)}, \ldots, X^{(n)}$ be the notation for a contravariant system, the results of this article may be written

$$(\mathbf{X}^\circ) = \mathbf{M}^{-1}\cdot\mathbf{X}^\circ, \quad \mathbf{X}^\circ = \mathbf{M}\cdot(\mathbf{X}^\circ), \quad (\mathbf{X}^\circ\mathbf{Y}^\circ) = \mathbf{M}^{-1}\mathbf{M}^{-1}{:}\mathbf{X}^\circ\mathbf{Y}^\circ, \text{ etc.}$$

If $\mathbf{A}$ be the matrix $||a_{ji}||$, and I the idemfactor we have further

$$\mathbf{A}\cdot\mathbf{A}^{-1} = \mathrm{I}, \quad \mathbf{A} = (\mathbf{A}){:}\mathbf{M}^{-1}\mathbf{M}^{-1}, \quad [(\mathbf{A}){:}\mathbf{M}^{-1}\mathbf{M}^{-1}]\cdot\mathbf{A}^{-1} = \mathrm{I}.$$

Now if $[\mathbf{C}{:}\mathbf{MN}]\cdot\mathbf{D} = \mathrm{I}$, then $\mathbf{C}\cdot[\mathbf{NM}{:}\mathbf{D}] = \mathrm{I}$. For

$$[\mathbf{C}{:}\mathbf{MN}]\cdot\mathbf{D} = \mathbf{M}_C\cdot\mathbf{C}\cdot\mathbf{N}\cdot\mathbf{D}, \qquad \mathbf{C}\cdot[\mathbf{NM}{:}\mathbf{D}] = \mathbf{C}\cdot\mathbf{N}\cdot\mathbf{D}\cdot\mathbf{M}_C,$$
$$\mathbf{M}_C\cdot\mathbf{C}\cdot\mathbf{N}\cdot\mathbf{D} = \mathrm{I}, \qquad \mathbf{C}\cdot\mathbf{N}\cdot\mathbf{D} = \mathbf{M}_C^{-1}, \qquad \mathbf{C}\cdot\mathbf{N}\cdot\mathbf{D}\cdot\mathbf{M}_C = \mathrm{I}.$$

Hence $\qquad [(\mathbf{A}){:}\mathbf{M}^{-1}\mathbf{M}^{-1}]\cdot\mathbf{A}^{-1} = (\mathbf{A})\cdot[\mathbf{M}^{-1}\mathbf{M}^{-1}{:}\mathbf{A}^{-1}] = \mathrm{I}$

or $\qquad (\mathbf{A}^{-1}) = \mathbf{M}^{-1}\mathbf{M}^{-1}{:}\mathbf{A}^{-1}, \qquad \mathbf{A}^{-1} = \mathbf{MM}{:}(\mathbf{A}^{-1}).$

This analysis could, of course, be carried out without the conversion of the double products into simple products; the conversion has been used because it may seem simpler to those familiar with matrices (products with a single opening only).

9. Dual systems.[14] Consider next the system of the first order

$$X^{(i)} = \Sigma_j a^{(ij)} X_j, \tag{15}$$

formed of a system X of the first order and the contravariant systems of the second order $a^{(ij)}$. We shall prove that this system $X^{(i)}$ is contravariant of the first order,— which will justify the use of upper indices. Carry out the transformation above; then, by (10) and (5),

$$(X^{(i)}) = \Sigma_j a^{(ij)}(X_j) = \Sigma_j \Sigma_{kl} \gamma_{ki} \gamma_{lj} a^{(kl)} \Sigma_p c_{pj} X_p.$$

The sum taken over j requires $l = p$. Hence by (12),

$$(X^{(i)}) = \Sigma_{kl} \gamma_{ki} a^{(kl)} X_l = \Sigma_k \gamma_{ki} X^{(k)},$$

and the theorem is proved. We thus have, associated with every covariant system, a contravariant system relative to the quadratic form (8).

If we proceed in a similar manner for systems of the second order, we may construct,

$$X^{(ij)} = \Sigma_{kl} a^{(ik)} a^{(jl)} X_{kl}. \tag{15'}$$

This likewise is seen to be a contravariant system. In general if we have a covariant system of order m, we may define a contravariant system of equal order by the equation

$$X^{(i_1 i_2 \ldots l_m)} = \Sigma_{j_1 j_2 \ldots j_m} a^{(i_1 j_1)} a^{(i_2 j_2)} \ldots a^{(i_m j_m)} X_{j_1 j_1 \ldots j_m}. \tag{15''}$$

Moreover this relation is reciprocal; for we may pass back to the original system by the formula,

$$X_{k_1 k_2 \ldots k_m} = \Sigma_{l_1 l_2 \ldots l_m} a_{l_1 k_1} a_{l_2 k_2} \cdots a_{l_m k_m} X^{(l_1 l_2 \cdots l_m)}. \tag{16}$$

To prove this we have merely to substitute from (15″) in (16), taking $j = k$, $i = l$, and use the fact that $\Sigma_j a^{(ij)} a_{lj} X_l = X_i$. Thus to every covariant system of order m corresponds a contravariant system of

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14 Ricci uses the term *reciprocal* systems in place of *dual* systems and there are advantages in this use; but we have preferred to reserve the term reciprocal for *sets of systems*, thereby following the notation of Gibbs in his vector analysis. The term dual suggests itself strongly in connection with a quadratic form.

like order and conversely to every contravariant system corresponds a covariant system of like order, with the systems occurring in dual pairs. In particular the systems a_{ij} and $a^{(ij)}$ are dual since,

$$a^{(ij)} = \Sigma_{kl}a^{(ik)}a^{(jl)}a_{kl}, \qquad a_{ij} = \Sigma_{kl}a_{ki}a_{lj}a^{(kl)}.$$

The results of this section may again be put in matrical form and gain in brevity. We set $\mathbf{X}^\circ = \mathbf{X}\cdot\mathbf{A}^{-1}$ or $\mathbf{X}^\circ = \mathbf{A}^{-1}\cdot\mathbf{X}$, it matters not which, since $\mathbf{A}$ is self conjugate. Then,

$$\mathbf{X}^\circ = \mathbf{X}\cdot\mathbf{A}^{-1} = [(\mathbf{X})\cdot\mathbf{M}^{-1}]\cdot[\mathbf{MM}:(\mathbf{A}^{-1})] = (\mathbf{X})\cdot(\mathbf{A}^{-1})\cdot\mathbf{M}_C = \mathbf{M}\cdot(\mathbf{X})\cdot(\mathbf{A}^{-1}),$$

which shows that $\mathbf{X}^\circ$ transforms contravariantly. The terms $X^{(ij)}$ may be treated as a symbolic product $\mathbf{X}^\circ\mathbf{Y}^\circ$ and the result is that $\mathbf{XY}:\mathbf{A}^{-1}\mathbf{A}^{-1}$ is contravariant, etc. The dual is obtained by writing $\mathbf{X} = \mathbf{X}^\circ\cdot\mathbf{A} = \mathbf{A}\cdot\mathbf{X}^\circ$.

10. Composition of systems. If we have any two systems X, Y, of order m, one covariant, the other contravariant, we may form

$$I = \Sigma_{i_1i_2\ldots i_m}X_{i_1i_2\ldots i_m}Y^{(i_1i_2\cdots i_m)}. \tag{17}$$

This system I contains only one element and is invariant. For

$$\Sigma_{i_1i_2\cdots i_m}X_{i_1i_2\cdots i_m}Y^{(i_1i_2\cdots i_m)} = \Sigma_{i_1i_2\ldots i_m}\Sigma_{j_1j_2\ldots j_m}\gamma_{i_1j_1}\gamma_{i_2j_2}\cdots \gamma_{i_mj_m}(X_{j_1j_2\ldots j_m})\,\Sigma_{k_1k_2\ldots k_n}c_{i_1k_1}c_{i_2k_2}\ldots c_{i_mk_m}(X^{(k_1k_2\cdots k_m)}),$$

and this reduces to $\Sigma_{j_1j_2\ldots j_m}(X_{j_1j_2\ldots j_m})\,(Y^{(j_1j_2\cdots j_m)})$; because when summed on i the right hand member gives something only when $k = j$, and then gives 1.

In like manner if we have a system X of order $m + p$ and a contravariant system Y of order m, we may get a system

$$Z_{i_1i_2\ldots i_p} = \Sigma_{j_1j_2\ldots j_m}X_{i_1i_2\ldots i_pj_1j_2\ldots j_m}Y^{(j_1j_2\cdots j_m)}, \tag{18}$$

of order p, which is covariant. By a similar definition,

$$Z^{(i_1i_2\cdots i_p)} = \Sigma_{j_1j_2\ldots j_m}X^{(i_1i_2\cdots i_pj_1j_2\cdots j_m)}Y_{j_1j_2\ldots j_m}. \tag{19}$$

We may combine a contravariant system of order $m + p$ with a

covariant system of order m to get a contravariant system of order p. The proofs of these facts are as above for the special case when the orders are equal.

The process by which covariant and contravariant systems are combined in (17), (18), (19) to obtain a system of order equal to the difference of the two orders is called *composition*.[15] (In the definition we have placed the common indices at the end. We may generalize the definition by distributing the indices in any way. Thus (15) and (16) may be considered as cases of composition of a system of order $2m$ with one of order m.)

Composition is very simple in matrical notation.

$$\mathbf{X}\cdot\mathbf{Y}^{\circ} = \mathbf{X}\cdot\mathbf{A}^{-1}\cdot\mathbf{Y} = \mathbf{XY}:\mathbf{A}^{-1}$$

is clearly invariant. If proof were needed, we could write

$$\mathbf{XY}:\mathbf{A}^{-1} = [(\mathbf{XY}):\mathbf{M}^{-1}\mathbf{M}^{-1}]:[\mathbf{MM}:(\mathbf{A}^{-1})] = (\mathbf{XY}):(\mathbf{A}^{-1}).$$

We have simply to take into account what elements the dots actually unite in the multiplications.

11. Mutually reciprocal[16] n-tuples. For any covariant system λ_r, consisting of n functions of the variables $x_1, x_2, \ldots x_n$, and the dual system $\lambda^{(r)}$ we have found by (15), (16) the relations

$$\lambda^{(r)} = \Sigma_s a^{(rs)}\lambda_s, \qquad \lambda_r = \Sigma_s a_{rs}\lambda^{(s)}. \tag{20}$$

Suppose that we have n systems ${}_1\lambda_r, {}_2\lambda_r, \ldots, {}_n\lambda_r$ and the corresponding dual systems ${}_1\lambda^{(r)}, {}_2\lambda^{(r)}, \ldots, {}_n\lambda^{(r)}$. The n systems ${}_i\lambda_r$ will be called *independent* if the determinant $|{}_i\lambda_r|$ does not vanish. As $|a_{rs}| \neq 0$, it follows at once that the dual systems are also independent. We may define contravariant systems ${}_i\lambda'^{(s)}$ in terms of ${}_i\lambda_r$ by the equations

$$\Sigma_{i}.{}_i\lambda'^{(s)}.{}_i\lambda_r = \epsilon_{rs}, \qquad \epsilon_{rs} = \begin{cases} 0, & r \neq s, \\ 1, & r = s. \end{cases} \tag{21}$$

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[15] Composition is a sort of inverse of multiplication in that the result of composition is to subtract the orders of the factors, whereas multiplication adds the orders. Composition itself may be regarded as a species of multiplication in the general sense in which Gibbs used the term, and has close analogies with regressive multiplication or with the inner product as defined by G. N. Lewis, *Proc. Amer. Acad. Arts Sci.*, **46**, 165–181 (1910), also (with Wilson) *Ibid.*, **48**, 389–507 (1912), especially § 29.

[16] See note 14.

For if s be held fixed and r take the values 1, 2,......, n, there are n equations (21) which are linear and non-homogeneous in the n variables ${}_i\lambda'^{(s)}$, and the equations are consistent because the determinant $|\,{}_i\lambda_r\,|$ of the coefficients does not vanish. If we replace ${}_i\lambda'^{(s)}$ and ${}_i\lambda_r$ by their values from (20) we have

$$\Sigma_i\Sigma_t a^{(st)}.{}_i\lambda'_t\Sigma_u a_{ru}.{}_i\lambda^{(u)} = \epsilon_{rs}.$$

Then

$$\Sigma_{pq}a^{(rq)}a_{sp}\Sigma_{itu}a^{(st)}.{}_i\lambda'_t a_{ru}.{}_i\lambda^{(u)} = \Sigma_{pq}a^{(rq)}a_{sp}\epsilon_{rs}.$$

Now by the reduction formula (13) the left hand side may twice be simplified. On the right hand side ϵ_{rs} vanishes unless $r = s$ and the double sum reduces to ϵ_{pq}. Hence

$$\Sigma_i.{}_i\lambda'_p.{}_i\lambda^{(q)} = \epsilon_{pq}. \qquad (21')$$

We see therefore that there is a reciprocal relation (21′) to (21) between the ${}_i\lambda_r$, ${}_i\lambda'_r$, ${}_i\lambda^{(r)}$, ${}_i\lambda'^{(r)}$.

The n systems ${}_i\lambda_r$ may be called a covariant n-tuple; the systems ${}_i\lambda^{(r)}$ the contravariant n-tuple; these are mutually dual in pairs. The set of n systems ${}_i\lambda'^{(r)}$ will be called *reciprocal* to the set ${}_i\lambda^{(r)}$ and the set ${}_i\lambda'_r$ reciprocal to the set ${}_i\lambda_r$. We may give a geometric analogy in support of this nomenclature. If we have a conic and three points P, Q, R, we may obtain the duals, the lines p, q, r. The points, however, determine three lines QR, RP, PQ and of these the duals are the points qr, rp, pq. The sets P, Q, R and qr, rp, pq are reciprocal; and similarly p, q, r and QR, RP, PQ. Another analogy would arise in spherical geometry where ABC and $A'B'C'$ are polar triangles; the sets A, B, C and A', B', C', being reciprocal. The use of reciprocal systems in vector analysis is prominent in the system of Gibbs, particularly for the solution of equations. If the n sets ${}_i\lambda_r$ form an *orthogonal* n-tuple, the reciprocal sets will be proportional to them — a *unit* orthogonal n-tuple is self-reciprocal (see infra, §13).

We may obtain in addition to the defining relations, the following between reciprocal n-tuples.

$$\Sigma_i.{}_i\lambda'_s.{}_i\lambda_t = a_{st}, \quad \Sigma_i.{}_i\lambda'^{(s)}.{}_i\lambda^{(t)} = a^{(st)}. \qquad (22)$$

These are proved in the usual fashion. If we compare the relations (21) which define the elements ${}_i\lambda'^{(s)}$ in terms of the elements ${}_i\lambda_r$ with

the relations $\Sigma_j c_{ir}\gamma_{is} = \epsilon_{rs}$ between the elements c_{ir} of any non-vanishing determinant $| c_{ir} |$ and the elements γ_{is} obtained by dividing the cofactor C_{is} by the determinant, we see at once that *the elements* ${}_i\lambda'^{(s)}$ *are the cofactors of* ${}_i\lambda_s$ *divided by* $| {}_i\lambda_r |$, — and similarly from (21′) the elements ${}_i\lambda'_p$ are the cofactors of ${}_i\lambda^{(p)}$ divided by $| {}_i\lambda^{(p)} |$. These relations are also reciprocal, i. e., the elements ${}_i\lambda^{(p)}$ and ${}_i\lambda_r$ are respectively the cofactors of ${}_i\lambda'_p$ and ${}_i\lambda'^{(r)}$ divided by the determinants $| {}_i\lambda'_p |$ and $| {}_i\lambda'^{(r)} |$. Hence by summing the other way, namely upon the index r we may get the relations

$$\Sigma_r . {}_i\lambda_r . {}_j\lambda'^{(r)} = \epsilon_{ij}, \quad \Sigma_r . {}_i\lambda'_r . {}_j\lambda^{(r)} = \epsilon_{ij}. \tag{22′}$$

12. A standard form for systems. If we have a contravariant n-tuple ${}_i\lambda^{(r)}$ and any covariant system X_r we may form by composition the n invariants

$$c_i = \Sigma_r X_r . {}_i\lambda^{(r)}.$$

These equations may be solved with the aid of the reciprocal n-tuple. For, by (21′),

$$\Sigma_i c_i . {}_i\lambda'_s = \Sigma_{ri} X_r . {}_i\lambda'_s . {}_i\lambda^{(r)} = \Sigma_r X_r \epsilon_{rs} = X_s.$$

Hence

$$X_s = \Sigma_i c_i . {}_i\lambda'_s. \tag{23}$$

Any system X_s is therefore representable as a linear function of ${}_i\lambda'_s$ with invariant coefficients. In like manner

$$X^{(s)} = \Sigma_i c_i . {}_i\lambda'^{(s)}, \quad c_i = \Sigma_r X^{(r)} {}_i\lambda_r. \tag{23′}$$

In general for systems of any order we may write

$$\begin{aligned} X_{r_1 r_2 \ldots r_k} &= \Sigma_{i_1 i_2 \ldots i_k} c_{i_1 i_2 \ldots i_k} . {}_{i_1}\lambda'_{r_1} . {}_{i_2}\lambda'_{r_2} \ldots {}_{i_k}\lambda'_{r_k}, \\ c_{i_1 i_2 \ldots i_k} &= \Sigma_{r_1 r_2 \ldots r_k} X_{r_1 r_2 \ldots r_k} . {}_{i_1}\lambda^{(r_1)} . {}_{i_2}\lambda^{(r_2)} \ldots {}_{i_k}\lambda^{(r_k)}, \end{aligned} \tag{23″}$$

and

$$\begin{aligned} X^{(r_1 r_2 \ldots r_k)} &= \Sigma_{i_1 i_2 \ldots i_k} c_{i_1 i_2 \ldots i_k} . {}_{i_1}\lambda'^{(r_1)} . {}_{i_2}\lambda'^{(r_2)} \ldots {}_{i_k}\lambda'^{(r_k)}, \\ c_{i_1 i_2 \ldots i_k} &= \Sigma_{r_1 r_2 \ldots r_k} X^{(r_1 r_2 \ldots r_k)} . {}_{i_1}\lambda_{r_1} . {}_{i_2}\lambda_{r_2} \ldots {}_{i_k}\lambda_{r_k}. \end{aligned} \tag{23‴}$$

Any system of order m is linearly dependent, with invariant coefficients, on the product system of the mth order made up of the λ''s.

As the λ's thus form *a basis* for the expression of systems in general we may set up readily the progressive product of Grassmann; for

$$\begin{vmatrix} X_r & X_s \\ Y_r & Y_s \end{vmatrix} = \Sigma_{ij} c_i c_j \begin{vmatrix} {}_i\lambda'_r & {}_j\lambda'_r \\ {}_i\lambda'_s & {}_j\lambda'_s \end{vmatrix}, \text{ etc.}$$

In the system of Grassmann the progressive product represents the space determined by the elements (a parallelogram in the case of two vectors); but the interpretation here is not so direct because the systems X_r of the first order are not components of a vector,— they have to be multiplied by certain factors to obtain components of a vector.[17] In like manner the terms $X_r Y_s - X_s Y_r$ are not components of a plane but may be converted into such by proper factors.

13. Orthogonal unit n-tuples. We may define orthogonality relative to a given quadratic form as in non-euclidean geometry. We shall now however take the form as differential, namely, as

$$ds^2 = \Sigma_{rs} a_{rs} dx_r dx_s .$$

Since the elements dx_r form a contravariant system (§4) a direction in space may be defined by any contravariant system $\lambda^{(r)}$ if we set up the simultaneous differential equations

$$\frac{dx_1}{\lambda^{(1)}} = \frac{dx_2}{\lambda^{(2)}} = \frac{dx_3}{\lambda^{(3)}} = = \ldots\ldots\ldots = \frac{dx_n}{\lambda^{(n)}}\,; \tag{24}$$

and it is in this way that the contravariant systems used above, and previously defined as contravariant systems, are associated with special directions.

If we have two systems ${}_i\lambda^{(r)}$, ${}_j\lambda^{(s)}$ we define as is customary in differ-

17 It is shown by Ricci and Levi-Civita (*Math. Ann.*, **60**) that if two dual systems of the first order X_r, $X^{(r)}$ are divided by $\sqrt{a_{rr}}$ and $\sqrt{a^{(rr)}}$, the resulting expressions $X_r/\sqrt{a_{rr}}$, $X^{(r)}/\sqrt{a^{(rr)}}$ may be regarded respectively as the orthogonal projections of one and the same vector upon the tangents to the coördinate lines x_r and upon the normals to the coördinate surfaces; whereas the expressions $X^{(r)}\sqrt{a_{rr}}$ and $X_r\sqrt{a^{(rr)}}$ represent respectively the components of the same vector along the same lines and the same normals. This process of rendering a system vectorial might be called vectorization and could be extended to vectors of higher order (Stufe).

ential or non-euclidean geometry, the angle between the directions by the formula

$$\cos\theta = \frac{\Sigma_{rs}a_{rs}\cdot{}_i\lambda^{(r)}\cdot{}_j\lambda^{(s)}}{\sqrt{\Sigma_{rs}a_{rs}\cdot{}_i\lambda^{(r)}\cdot{}_i\lambda^{(s)}}\ \sqrt{\Sigma_{rs}a_{rs}\cdot{}_j\lambda^{(r)}\cdot{}_j\lambda^{(s)}}}, \tag{25}$$

where the λ's are proportional to the differentials by (24). The condition of orthogonality for the two directions ${}_i\lambda^{(r)}$, ${}_j\lambda^{(s)}$ is therefore

$$\Sigma_{rs}a_{rs}\cdot{}_i\lambda^{(r)}\cdot{}_j\lambda^{(s)} = 0.$$

This may be written $\Sigma_s\cdot{}_i\lambda_s\cdot{}_j\lambda^{(s)} = 0$, by using the covariant system.[18] Our results may be simplified by considering the systems ${}_i\lambda^{(r)}$, ${}_j\lambda^{(s)}$ in (24) as first multiplied by such a factor that the radicals in (25) reduce to unity, that is, so that $\Sigma_{rs}a_{rs}\cdot{}_i\lambda^{(r)}\cdot{}_i\lambda^{(s)} = 1$. Such a system may be called a *unit* system. The conditions for a unit orthogonal n-tuple are therefore,

$$\Sigma_s\cdot{}_i\lambda_s\cdot{}_j\lambda^{(s)} = \epsilon_{ij}. \tag{26}$$

Now if we multiply (26) by ${}_j\lambda'_r$, sum over j, and apply (21′) we have ${}_i\lambda_r = {}_i\lambda'_r$. and in like manner we should have ${}_i\lambda^{(r)} = {}_i\lambda'^{(r)}$. Hence for a unit orthogonal n-tuple the reciprocal and given sets of systems are identical. This gives from (21) the relation

$$\Sigma_i\cdot{}_i\lambda_r\cdot{}_i\lambda^{(s)} = \epsilon_{rs}, \tag{27}$$

in addition to (26) for unit orthogonal n-tuples. The relations (26) and (27) are like those connecting the directions cosines of an orthogonal set in ordinary space. We may get from (22) the relations

$$\Sigma_{i_i i}\lambda_r\cdot{}_i\lambda_s = a_{rs}, \quad \Sigma_i\cdot{}_i\lambda^{(r)}{}_i\lambda^{(s)} = a^{(rs)} \tag{28}$$

14. Transformations of variables. Though the forms in which we are interested are differential and the transformations of variable arbitrary,

$$x_1 = x_1(y_1, y_2, \ldots, y_n), \ldots\ldots, \quad x_n = x_n(y_1, y_2, \ldots, y_n),$$

18 If we compare this condition of perpendicularity with (22′) we see that the direction ${}_i\lambda'^{(r)}$ is perpendicular to the direction ${}_j\lambda^{(r)}$ for all values of i except $i = j$. If we consider all the directions linearly derived from ${}_i\lambda^{(r)}$, $i \neq j$, we find that they determine the $(n-1)$-space perpendicular to ${}_j\lambda'^{(r)}$.

the transformation of the differentials is linear;— thus

$$dx_i = \Sigma_j \frac{\partial x_i}{\partial y_j} dy_j = \Sigma_j c_{ij} dy_j, \qquad c_{ij} = \frac{\partial x_i}{\partial y_j},$$

with the difference over the algebraic theory that the coefficients c_{ij} are variable. As the work done to this point does not involve derivatives of the c's or in any way depend on their constancy, the whole work remains valid. As the particular relations

$$c_{ij} = \frac{\partial x_i}{\partial y_j}, \qquad \gamma_{ji} = \frac{\partial y_i}{\partial x_j}$$

now hold we may define covariant and contravariant systems of order k as those for which

$$X_{i_1 i_2 \ldots i_k} = \Sigma_{j_1 j_2 \ldots j_k} (X_{j_1 j_2 \ldots j_k}) \frac{\partial y_{j_1}}{\partial x_{i_1}} \frac{\partial y_{j_2}}{\partial x_{i_2}} \cdots \frac{\partial y_{j_k}}{\partial x_{i_k}}, \tag{29}$$

$$(X_{i_1 i_2 \ldots i_k}) = \Sigma_{j_1 j_2 \ldots j_k} X_{j_1 j_2 \ldots j_k} \frac{\partial x_{j_1}}{\partial y_{i_1}} \frac{\partial x_{j_2}}{\partial y_{i_2}} \cdots \frac{\partial x_{j_k}}{\partial y_{i_k}}, \tag{29'}$$

$$X^{(i_1 i_2 \cdots i_k)} = \Sigma_{j_1 j_2 \ldots j_k} (X^{(j_1 j_2 \cdots j_k)}) \frac{\partial x_{i_1}}{\partial y_{j_1}} \frac{\partial x_{i_2}}{\partial y_{j_2}} \cdots \frac{\partial x_{i_k}}{\partial y_{j_k}}, \tag{30}$$

$$(X^{(i_1 i_2 \cdots j_k)}) = \Sigma_{j_1 j_2 \ldots j_k} X^{(j_1 j_2 \cdots j_k)} \frac{\partial y_{i_1}}{\partial x_{j_1}} \frac{\partial y_{i_2}}{\partial x_{j_2}} \cdots \frac{\partial y_{i_k}}{\partial x_{j_k}}. \tag{30'}$$

If we have a function of the variables, the derivatives $f_i = \partial f / \partial x_i$ form a system of the first order. We know that,

$$\frac{\partial f}{\partial x_i} = \Sigma_j \frac{\partial f}{\partial y_j} \frac{\partial y_j}{\partial x_i} = \Sigma_j \left(\frac{\partial f}{\partial x_j}\right) \frac{\partial y_j}{\partial x_i},$$

since $\partial f/\partial y_j = (\partial f/\partial x_j)$ by definition. Hence we see that the first derivatives of any function (system of order 0) form a covariant system of order 1.

If we have a general covariant system X_i of the first order, the derivatives of the elements of the system with respect to the variables,

$X_{ij} = \partial X_j / \partial x_j$, would form a system of the second order. Let us consider the transformation of this system.

$$X_{ij} = \frac{\partial X_i}{\partial x_j} = \frac{\partial}{\partial x_j} \Sigma_k \gamma_{ik} (X_k) = \Sigma_k \gamma_{ik} \frac{\partial (X_k)}{\partial x_j} + \Sigma_k (X_k) \frac{\partial \gamma_{ik}}{\partial x_j}$$
$$= \Sigma_k \gamma_{ik} \Sigma_l \frac{\partial (X_k)}{\partial y_l} \frac{\partial y_l}{\partial x_j} + \Sigma_k (X_k) \frac{\partial \gamma_{ik}}{\partial x_j}$$
$$= \Sigma \gamma_{ik} \gamma_{jl} (X_{kl}) + \Sigma_k (X_k) \frac{\partial^2 y_k}{\partial x_i \partial x_j}. \tag{31}$$

If it were not for the second term, the transformation would be covariant, but the presence of this term shows that the derivatives of a covariant system of the first order do not form a covariant system of the second order.

The same is true for covariant systems of any order,— their derivatives do not form a covariant system. For instance in the case of a covariant system of the second order X_{rs}, by a similar transformation,

$$\frac{\partial X_{rs}}{\partial x_t} = \Sigma_{ijk} \gamma_{ri} \gamma_{sj} \gamma_{tk} \frac{\partial (X_{ij})}{\partial y_k} + \Sigma_{ij} (X_{ij}) \left[\frac{\partial^2 y_i}{\partial x_r \partial x_t} \frac{\partial y_j}{\partial x_s} + \frac{\partial^2 y_j}{\partial x_s \partial x_t} \frac{\partial y_i}{\partial x_r} \right], \tag{31'}$$

where $\qquad \partial y_i / \partial x_s = \gamma_{si}$ and $\partial^2 y_i / \partial x_r \partial x_t = \partial \gamma_{ri} / \partial x_t$.

The fundamental relation $dx_i = \Sigma_i c_{ij} dy_j$ may be written in matrical notation as $d\mathbf{x} = d\mathbf{y} \cdot \nabla_y \mathbf{x}$. It follows that $\mathbf{M}_C = \nabla_y \mathbf{x}$. We may also write $d\mathbf{y} = d\mathbf{x} \cdot \nabla_x \mathbf{y}$. Hence $\nabla_x \mathbf{y}$ and $\nabla_y \mathbf{x}$ are reciprocals. The relations (29) and (30) may be written as

$$\begin{aligned}
\mathbf{X} &= \nabla_x \mathbf{y} \cdot (\mathbf{X}), & \mathbf{XY} &= \nabla_x \mathbf{y} \nabla_x \mathbf{y} : (\mathbf{XY}), \\
(\mathbf{X}) &= \nabla_y \mathbf{x} \cdot \mathbf{X}, & (\mathbf{XY}) &= \nabla_y \mathbf{x} \nabla_y \mathbf{x} : \mathbf{XY}, \\
\mathbf{X}^\circ &= (\mathbf{X}^\circ) \cdot \nabla_y \mathbf{x}, & \mathbf{X}^\circ \mathbf{Y}^\circ &= (\mathbf{X}^\circ \mathbf{Y}^\circ) : \nabla_y \mathbf{x} \nabla_y \mathbf{x}, \\
(\mathbf{X}^\circ) &= \mathbf{X}^\circ \cdot \nabla_x \mathbf{y}, & (\mathbf{X}^\circ \mathbf{Y}^\circ) &= \mathbf{X}^\circ \mathbf{Y}^\circ : \nabla_x \mathbf{y} \nabla_x \mathbf{y},
\end{aligned}$$

and so for systems of any order.

The differentiation of a system of the zeroth order f is accomplished as:

$$df = d(f), \qquad d\mathbf{x} \cdot \nabla_x f = d\mathbf{y} \cdot \nabla_y (f),$$
$$d\mathbf{x} \cdot \nabla_x f = d\mathbf{x} \cdot \nabla_x \mathbf{y} \cdot \nabla_y (f), \qquad \nabla f = \nabla_x \mathbf{y} \cdot (\nabla f).$$

This shows that ∇f is covariant of order 1. To differentiate a system $\mathbf{X}$ of order 1 we have

$$
\begin{aligned}
d\mathbf{X} &= \nabla_x \mathbf{y} \cdot d(\mathbf{X}) + d\nabla_x \mathbf{y} \cdot (\mathbf{X}) \\
d\mathbf{x} \cdot \nabla \mathbf{X} &= \nabla_x \mathbf{y} \cdot [d\mathbf{x} \cdot \nabla_x \mathbf{y} \cdot \nabla_y (\mathbf{X})] + d\mathbf{x} \cdot \nabla_x \nabla_x \mathbf{y} \cdot (\mathbf{X}) \\
\nabla \mathbf{X} &= \nabla_x \mathbf{y} \nabla_x \mathbf{y} : (\nabla \mathbf{X}) + \nabla_x \nabla_x \mathbf{y} \cdot (\mathbf{X}).
\end{aligned}
$$

15. Solution for the second derivatives.[19] As we are working with a fundamental adjoined quadratic form $\Sigma a_{rs} dx_r dx_s$, we regard the a_{rs} and their derivatives as known. We may then write

$$
\frac{\partial a_{rs}}{\partial x_t} = \Sigma_{ijk} \gamma_{ri} \gamma_{sj} \gamma_{tk} \frac{\partial (a_{ij})}{\partial y_k} + \Sigma_{ij} (a_{ij}) \left[\frac{\partial^2 y_i}{\partial x_r \partial x_t} \gamma_{sj} + \frac{\partial^2 y_j}{\partial x_s \partial x_t} \gamma_{ri} \right]
$$

and solve the six equations obtained by permuting r, s, t for the six derivatives $\partial^2 y_i / \partial x_r \partial x_t$ as unknowns. We have

$$
\frac{\partial a_{rt}}{\partial x_s} = \Sigma_{ij} \gamma_{ri} \gamma_{tj} \gamma_{sk} \frac{\partial (a_{ij})}{\partial y_k} + \Sigma_{ij} (a_{ij}) \left[\frac{\partial^2 y_i}{\partial x_r \partial x_s} \gamma_{tj} + \frac{\partial^2 y_j}{\partial x_s \partial x_t} \gamma_{ri} \right],
$$

$$
\frac{\partial a_{ts}}{\partial x_r} = \Sigma_{ijk} \gamma_{ti} \gamma_{sj} \gamma_{rk} \frac{\partial (a_{ij})}{\partial y_k} + \Sigma_{ij} (a_{ij}) \left[\frac{\partial^2 y_i}{\partial x_t \partial x_r} \gamma_{sj} + \frac{\partial^2 y_j}{\partial x_s \partial x_r} \gamma_{ti} \right].
$$

Hence

$$
\begin{aligned}
\frac{\partial a_{rt}}{\partial x_s} + \frac{\partial a_{ts}}{\partial x_r} - \frac{\partial a_{rs}}{\partial x_t} &= \Sigma_{ijk} \left[\gamma_{ri} \gamma_{tj} \gamma_{sk} + \gamma_{ti} \gamma_{sj} \gamma_{rk} - \gamma_{ri} \gamma_{sj} \gamma_{tk} \right] \frac{\partial (a_{ij})}{\partial y_k} \\
&\quad + \Sigma_{ij} (a_{ij}) \left[\frac{\partial^2 y_i}{\partial x_r \partial x_s} \gamma_{tj} + \frac{\partial^2 y_j}{\partial x_s \partial x_r} \gamma_{ti} \right].
\end{aligned}
$$

But as $(a_{ij}) = (a_{ji})$ we have,

$$
\Sigma_{ij} (a_{ij}) \frac{\partial^2 y_i}{\partial x_s \partial x_r} \gamma_{tj} = \Sigma_{ij} (a_{ij}) \frac{\partial^2 y_j}{\partial x_s \partial x_r} \gamma_{ti}
$$

---

[19] The solution for the second derivatives, though cumbersome, is exceedingly important for it is through this substitution that the Christoffel symbols actually arise (see Christoffel, *Gesammelte Werke*, or *Crelle J. Math.*, **70**, 46.) The method followed in so many books, viz., to write down the Christoffel symbols without any preliminaries seems decidedly artificial. We may point out that *when the analysis is carried on in matrical notation, as below, the elimination suggests itself much more readily than when we have so many subscripts and summation signs to manipulate as in the ordinary derivation.*

and the last bracket becomes a single term repeated. Moreover the first bracket may be changed by interchanging the indices i, j, k. For,

$$\Sigma_{ijk}\gamma_{ri}\gamma_{tj}\gamma_{sk}\frac{\partial(a_{ij})}{\partial y_k} = \Sigma_{ijk}\gamma_{ri}\gamma_{sj}\gamma_{tk}\frac{\partial(a_{ik})}{\partial y_j}$$

since in either case the summation is over all values of j and k. Hence,

$$\frac{1}{2}\left[\frac{\partial a_{rt}}{\partial x_s} + \frac{\partial a_{ts}}{\partial x_r} - \frac{\partial a_{rs}}{\partial x_t}\right] = \tfrac{1}{2}\Sigma_{ijk}\gamma_{ri}\gamma_{sj}\gamma_{tk}\left[\frac{\partial(a_{ik})}{\partial y_j} + \frac{\partial(a_{kj})}{\partial y_i} - \frac{\partial(a_{ij})}{\partial y_k}\right]$$
$$+ \Sigma_{ij}(a_{ij})\frac{\partial^2 y_i}{\partial x_s \partial x_r}\gamma_{tj}.$$

This somewhat cumbrous form may be simplified by introducing the notation of the Christoffel symbols,

$$\begin{bmatrix} r\,s \\ t \end{bmatrix} = \frac{1}{2}\left[\frac{\partial a_{rt}}{\partial x_s} + \frac{\partial a_{ts}}{\partial x_r} - \frac{\partial a_{rs}}{\partial x_t}\right]. \tag{32}$$

The above expression then becomes

$$\begin{bmatrix} s\,r \\ t \end{bmatrix} = \Sigma_{ijk}\gamma_{ri}\gamma_{sj}\gamma_{tk}\left(\begin{bmatrix} i\;j \\ k \end{bmatrix}\right) + \Sigma_{ij}(a_{ij})\frac{\partial^2 y_i}{\partial x_s \partial x_r}\gamma_{tj}.$$

To complete the solution for the second derivatives, multiply by c_{tl} and sum over t. Then

$$\Sigma_t c_{tl}\begin{bmatrix} r\,s \\ t \end{bmatrix} = \Sigma_{ij}\gamma_{ri}\gamma_{sj}\left(\begin{bmatrix} i\,j \\ l \end{bmatrix}\right) + \Sigma_i(a_{il})\ \frac{\partial^2 y_i}{\partial x_s \partial x_r}.$$

Next multiply by $(a^{(ml)})$ and sum over l. Then

$$\Sigma_{tl}c_{tl}(a^{(ml)})\begin{bmatrix} r\,s \\ t \end{bmatrix} = \Sigma_{ijl}\gamma_{ri}\gamma_{sj}(a^{(ml)})\left[\begin{pmatrix} i\,j \\ l \end{pmatrix}\right] + \frac{\partial^2 y_m}{\partial x_r \partial x_s},$$

and hence finally we have the expression

$$\frac{\partial^2 y_m}{\partial x_r \partial x_s} = \Sigma_{tl}c_{tl}(a^{(ml)})\begin{bmatrix} r\,s \\ t \end{bmatrix} - \Sigma_{ijl}\gamma_{ri}\gamma_{sj}(a^{(ml)})\left(\begin{bmatrix} i\,j \\ l \end{bmatrix}\right). \tag{33}$$

To differentiate the matrix **A** of the coefficients a_{ij} and obtain an expression for the second derivative we proceed as follows.[20] As **A** is self conjugate we may write **A** symbolically for the analytic work as **XX**. Then as

$$\mathbf{A} = \nabla \mathbf{y} \nabla \mathbf{y} : (\mathbf{A}), \quad \mathbf{XX} = \nabla \mathbf{y} \nabla \mathbf{y} : (\mathbf{XX}).$$

If we use subscripts to indicate the variables to which the differentiations apply, we have

$$\mathbf{XX} = \nabla_1 \nabla_2 \mathbf{y}_1 \cdot (\mathbf{X}) \mathbf{y}_2 \cdot (\mathbf{X}).$$

The symbols $\mathbf{y}_1 \cdot (\mathbf{X})$ and $\mathbf{y}_2 \cdot (\mathbf{X})$ are scalar, ∇_1 and ∇_2 extensive magnitudes. Now

$$\nabla \mathbf{XX} = [\nabla_1 \nabla_1 \nabla_2 + \nabla_2 \nabla_1 \nabla_2 + \nabla\, \nabla_1 \nabla_2]\, \mathbf{y}_1 \cdot (\mathbf{X}) \mathbf{y}_2 \cdot (\mathbf{X}),$$

---

20 Knoblauch in the preface of his *Grundlagen der Differential Geometrie* (1913) lays stress on the necessity of some operation such as his geometric differentiation to illuminate the formulas of differential geometry and while acknowledging the importance of Ricci's work, especially the *Lezioni*, complains that instead of using geometric derivatives he for the most part uses their "Coeffizienten System." A part of this difficulty is obviated by the use of the notations of multiple algebra as here employed by us and more of it by the large use of vectors that we make later in the work. By the combination of these two elements the analysis can be kept measurably simple and interpretable.

When discussing methods in differential geometry we must not omit that of Maschke; of which an account may be found in the following articles: Maschke, *Trans. Amer. Math. Soc.*, **1**, 197–204, *Ibid.*, **4**, 445–469, *Ibid.*, **7**, 69–80, 81–93; A. W. Smith, *Ibid.*, **7**, 33–60; Ingold, *Ibid.*, **11**, 449–474. For the actual use of the method in the theory of surfaces Smith's article is by far the most important of these references. One may say somewhat epigrammatically that Maschke's method contrasts with Ricci's in much the same way that the Clebsch-Aronhold method contrasts with Grassmann's. The fundamental element in Maschke's work is a *symbolic* treatment of the quadratic differential form. The reason that we have not used this method is because we have a natural preference for the non-symbolic method which is not overborne, for the simple work that we have in hand, by the gain in simplicity of operation of the symbolic method. In particular in regard to the present question of the solution for the second derivatives and the introduction of the Christoffel symbols we may observe that for Maschke's interpretation of $f_i f_{kl}$ as a Christoffel symbol it is necessary to assume that the symbols f_{kl} and f_{lk} are equal. Under this assumption $f_i f_{kl}$ appears as a Christoffel symbol and its appearance in this form may be taken as a justification for considering the symbols f_{lk} and f_{kl} as equal (for two of the indices in the Christoffel symbol are commutative). A very natural way to arrive at the Christoffel symbols is by Shaw's method (loc. cit., note 1) in which the symbols all have a geometric meaning; but unfortunately in order to follow this method we have to regard the surface as immersed in a space so that $ds^2 = d\mathbf{r} \cdot d\mathbf{r}$, and for theoretical purposes it is preferable at this stage to remain entirely upon the surface.

where the symbol ∇ applies to the variables $\mathbf{X}$ and $(\mathbf{X})$. Next,

$$\mathbf{X}\nabla\mathbf{X} = [\nabla_1\nabla_1\nabla_2 + \nabla_1\nabla_2\nabla_2 + \nabla_1\nabla\,\nabla_2]\,\mathbf{y}_1\cdot(\mathbf{X})\mathbf{y}_2\cdot(\mathbf{X})$$

may be obtained by interchanging the first and second extensive magnitudes. And

$$\mathbf{X}\mathbf{X}\nabla = [\nabla_1\nabla_2\nabla_1 + \nabla_1\nabla_2\nabla_2 + \nabla_1\nabla_2\nabla]\,\mathbf{y}_1\cdot(\mathbf{X})\mathbf{y}_2\cdot(\mathbf{X})$$

follows from another interchange. Now

$$\begin{aligned}\nabla\mathbf{X}\mathbf{X} + \mathbf{X}\nabla\mathbf{X} - \mathbf{X}\mathbf{X}\nabla = {} & 2\nabla_1\nabla_1\nabla_2\mathbf{y}_1\cdot(\mathbf{X})\mathbf{y}_2\cdot(\mathbf{X}) \\ & + [\nabla\,\nabla_1\nabla_2 + \nabla_1\nabla\,\nabla_2 - \nabla_1\nabla_2\nabla]\mathbf{y}_1\cdot(\mathbf{X})\mathbf{y}_2\cdot(\mathbf{X}),\end{aligned}$$

because $\nabla_2\nabla_1\nabla_2\mathbf{y}_1\cdot(\mathbf{X})\mathbf{y}_2\cdot(\mathbf{X})$ and $\nabla_1\nabla_2\nabla_1\mathbf{y}_1\cdot(\mathbf{X})\mathbf{y}_2\cdot(\mathbf{X})$ are the same. Thus far ∇ has denoted differentiation by $\mathbf{x}$. But $\nabla_x = \nabla_x\mathbf{y}\cdot\nabla_y = \nabla\mathbf{y}\cdot(\nabla)$. The terms in the bracket on the right may therefore be written

$$\nabla\,\nabla_1\nabla_2 = \nabla\mathbf{y}\cdot(\nabla)\nabla_1\nabla_2\mathbf{y}_1\cdot(\mathbf{X})\mathbf{y}_2\cdot(\mathbf{X}) = \nabla\mathbf{y}\nabla\mathbf{y}\nabla\mathbf{y}\,\vdots\,(\nabla\mathbf{X}\mathbf{X}),$$

and so on; hence

$$\begin{aligned}\nabla\mathbf{X}\mathbf{X} + \mathbf{X}\nabla\mathbf{X} - \mathbf{X}\mathbf{X}\nabla = 2\nabla\nabla\mathbf{y}\cdot(\mathbf{X}\mathbf{X})\cdot\nabla\mathbf{y}_C + \nabla\mathbf{y}\nabla\mathbf{y}\colon [(\nabla\mathbf{X}\mathbf{X} + \mathbf{X}\nabla\mathbf{X} \\ - \mathbf{X}\mathbf{X}\nabla)]\cdot\nabla\mathbf{y}_C\end{aligned}$$

or

$$\begin{aligned}2\nabla\nabla\mathbf{y} = [\nabla\mathbf{X}\mathbf{X} + \mathbf{X}\nabla\mathbf{X} - \mathbf{X}\mathbf{X}\nabla]\cdot\nabla_y\mathbf{x}_C\cdot(\mathbf{A}^{-1}) + \nabla\mathbf{y}\nabla\mathbf{y}\colon \\ [(\nabla\mathbf{X}\mathbf{X} + \mathbf{X}\nabla\mathbf{X} - \mathbf{X}\mathbf{X}\nabla)]\cdot\mathbf{A}^{-1}).\end{aligned}$$

The elements of this triadic are (compare 33)

$$\begin{aligned}2\,\frac{\partial^2 y_t}{\partial x_r\partial x_s} &= \Sigma_{pq}\left[\frac{\partial a_{rq}}{\partial x_r} + \frac{\partial a_{rq}}{\partial x_s} - \frac{\partial a_{rs}}{\partial x_q}\right] c_{qp}(a^{(pt)}) \\ &\quad + \Sigma_{pqn}\left[\frac{\partial(a_{qn})}{\partial y_p} + \frac{\partial(a_{pn})}{\partial y_q} - \frac{\partial(a_{pq})}{\partial y_r}\right]\gamma_{rp}\gamma_{sq}(a^{(nt)}) \\ &= 2\,\Sigma_{pq}\gamma_{pt}a^{(qp)}\begin{bmatrix} r\ s \\ q \end{bmatrix} - 2\,\Sigma_{pqn}\gamma_{rp}\gamma_{sq}(a^{(nt)})\left(\begin{bmatrix} p\ q \\ n \end{bmatrix}\right).\end{aligned}$$

16. Covariant differentiation of a simple system. Let us now substitute from (33) for the second derivatives in (31). Then

$$\frac{\partial X_i}{\partial x_j} = \Sigma_{kl}\gamma_{ik}\gamma_{jl}\frac{\partial(X_k)}{\partial y_l} + \Sigma_k(X_k)\Sigma_{tl}c_{tl}(a^{(kl)})\begin{bmatrix} i\,j \\ t \end{bmatrix}$$
$$- \Sigma_k(X_k)\Sigma_{rsl}\gamma_{ir}\gamma_{js}(a^{(kl)})\begin{bmatrix} r\,j \\ l \end{bmatrix}$$

or

$$\frac{\partial X_i}{\partial x_j} - \Sigma_k(X_k)\Sigma_{tl}c_{tl}(a^{(kl)})\begin{bmatrix} i\,j \\ t \end{bmatrix} = \Sigma_{kl}\gamma_{ik}\gamma_{jl}\frac{\partial(X_k)}{\partial y_l}$$
$$- \Sigma_t(X_t)\Sigma_{klp}\gamma_{ik}\gamma_{jl}(a^{(tp)})\left(\begin{bmatrix} k\,l \\ p \end{bmatrix}\right).$$

Now the presence of the multipliers γ_{ik}, γ_{jl} on the right makes it look as though the left might be a covariant of the second order and if we replace (X_k) by its value $\Sigma_m X_m c_{mk}$ and $(a^{(kl)})$ by its value $\Sigma_{pq}a^{(pq)}\gamma_{pk}\gamma_{ql}$, we find that

$$\Sigma_k(X_k)\Sigma_{tl}c_{tl}(a^{(kl)})\begin{bmatrix} i\,j \\ t \end{bmatrix} = \Sigma_{ktlmpq}X_m c_{mk}c_{tl}\gamma_{pk}\gamma_{ql}a^{(pq)}\begin{bmatrix} i\,j \\ t \end{bmatrix}$$
$$= \Sigma_{pq}X_p a^{(pq)}\begin{bmatrix} i\,j \\ q \end{bmatrix}.$$

Hence we have

$$\frac{\partial X_i}{\partial x_j} - \Sigma_{pq}X_p a^{(pq)}\begin{bmatrix} i\,j \\ q \end{bmatrix} = \Sigma_{kl}\gamma_{ik}\gamma_{jl}\left\{\frac{\partial(X_k)}{\partial y_l} - \Sigma_{tp}(X_t)\,(a^{(tp)})\left(\begin{bmatrix} k\,l \\ p \end{bmatrix}\right)\right\}.$$

We therefore write, as the *covariant derivative* of the system X_i of order 1, the covariant system of order 2,

$$X_{ij} = \frac{\partial X_i}{\partial x_j} - \Sigma_{pq}X_p a^{(pq)}\begin{bmatrix} i\,j \\ q \end{bmatrix}. \tag{34}$$

This system may be written a little more simply by introducing the Christoffel symbols of the second kind,

$$\Sigma_q a^{(pq)}\begin{bmatrix} i\,j \\ q \end{bmatrix} = \left\{\begin{matrix} i\,j \\ p \end{matrix}\right\}. \tag{35}$$

This is a sort of partial dual of the symbol of the first kind. Then

$$X_{ij} = \frac{\partial X_i}{\partial x_j} - \Sigma_p X_p \begin{Bmatrix} i\ j \\ p \end{Bmatrix}. \tag{34'}$$

The partial derivatives $\partial X_i/\partial x_j$ are expressible in terms of the derived system X_{ij} as

$$\frac{\partial X_i}{\partial x_j} = X_{ij} + \Sigma_p X_p \begin{Bmatrix} i\ j \\ p \end{Bmatrix}. \tag{34''}$$

If now we take the expression $\nabla\mathbf{X} = \nabla\mathbf{y}\nabla\mathbf{y}{:}(\nabla\mathbf{X}) + \nabla\nabla\mathbf{y}\cdot(\mathbf{X})$, and substitute for $\nabla\nabla\mathbf{y}$, we have

$$\nabla\mathbf{X} = \nabla\mathbf{y}\nabla\mathbf{y}{:}(\nabla\mathbf{X}) + \tfrac{1}{2}\begin{bmatrix} \mathbf{X}\ \mathbf{X} \\ \nabla \end{bmatrix}(\mathbf{A}^{-1})\cdot\nabla\mathbf{y}\cdot(\mathbf{X}) - \tfrac{1}{2}\nabla\mathbf{y}\nabla\mathbf{y}{:}\left(\begin{bmatrix} \mathbf{X}\ \mathbf{X} \\ \nabla \end{bmatrix}\right)\cdot(\mathbf{A}^{-1})\cdot(\mathbf{X})$$

or

$$\nabla\mathbf{X} - \tfrac{1}{2}\begin{bmatrix} \mathbf{X}\ \mathbf{X} \\ \nabla \end{bmatrix}\cdot\mathbf{A}^{-1}\cdot\mathbf{X} = \nabla\mathbf{y}\nabla\mathbf{y}{:}\left\{(\nabla\mathbf{X}) - \tfrac{1}{2}\left(\begin{bmatrix} \mathbf{X}\ \mathbf{X} \\ \nabla \end{bmatrix}\right)\cdot(\mathbf{A}^{-1})\cdot(\mathbf{X})\right\}.$$

If this be expanded we have, as before,

$$\frac{\partial X_i}{\partial x_j} - \Sigma_{pq}\begin{bmatrix} i\ j \\ q \end{bmatrix} a^{(qp)}X_p = \sum{}_{mn}\gamma_{im}\gamma_{in}\left\{\frac{\partial X_n}{\partial x_m} - \Sigma_{pq}\left(\begin{bmatrix} n\ m \\ q \end{bmatrix}\right)(a^{(qp)})X_p\right\}.$$

17. Covariant differentiation of systems of higher order. To find the covariant derivative of a system of the second order we must substitute from (33) for the second derivatives in (31′) and reduce. There are two terms containing second derivatives. We have

$$\Sigma_{ij}(X_{ij})\frac{\partial^2 y_i}{\partial x_r \partial x_t}\gamma_{sj} = \Sigma_{ij}(X_{ij})\gamma_{sj}\Sigma_{pl}c_{pl}(a^{(il)})\begin{bmatrix} r\ t \\ p \end{bmatrix} - \Sigma_{ij}(X_{ij})\gamma_{sj}\Sigma_{pql}\gamma_{rp}\gamma_{tq}(a^{(il)})\left(\begin{bmatrix} p\ q \\ l \end{bmatrix}\right).$$

The second term here is

$$\Sigma_{jpq}\gamma_{sj}\gamma_{rp}\gamma_{tq}\Sigma_{il}(X_{ij})(a^{(il)})\left(\begin{bmatrix} p & q \\ & l \end{bmatrix}\right) = \Sigma_{ijpq}\gamma_{sj}\gamma_{rp}\gamma_{tq}(X_{ij})\left(\begin{Bmatrix} p & q \\ & i \end{Bmatrix}\right).$$

The first term may be written as

$$\Sigma_{ijuvmnpql}X_{uv}c_{ui}c_{vj}c_{pl}\gamma_{sj}a^{(mn)}\gamma_{mi}\gamma_{nl}\begin{bmatrix} r & t \\ & p \end{bmatrix} =$$

$$\Sigma_{mn}X_{ms}a^{(mn)}\begin{bmatrix} r & t \\ & n \end{bmatrix} = \Sigma_m X_{ms}\begin{Bmatrix} r & t \\ & m \end{Bmatrix}.$$

Hence

$$\Sigma_{ij}(X_{ij})\frac{\partial^2 y_i}{\partial x_r \partial x_t}\gamma_{sj} = \Sigma_m\left[X_{ms}\begin{Bmatrix} r & t \\ & m \end{Bmatrix} - \Sigma_{jpq}\gamma_{sj}\gamma_{rp}\gamma_{tq}(X_{mj})\left(\begin{Bmatrix} p & q \\ & m \end{Bmatrix}\right)\right],$$

and in like manner,

$$\Sigma_{ij}(X_{ij})\frac{\partial^2 y_j}{\partial x_s \partial x_t}\gamma_{ri} = \Sigma_m\left[X_{rm}\begin{Bmatrix} s & t \\ & m \end{Bmatrix} - \Sigma_{jpq}\gamma_{rj}\gamma_{sp}\gamma_{tq}(X_{jm})\left(\begin{Bmatrix} p & q \\ & m \end{Bmatrix}\right)\right].$$

Hence

$$X_{rst} = \frac{\partial X_{rs}}{\partial x_t} - \Sigma_m\left[X_{ms}\begin{Bmatrix} r & t \\ & m \end{Bmatrix} + X_{rm}\begin{Bmatrix} s & t \\ & m \end{Bmatrix}\right] \tag{36}$$

transforms covariantly as of order three.

We may generalize to the next higher order as

$$X_{rstu} = \frac{\partial X_{rst}}{\partial x_u} - \Sigma_m\left[X_{mst}\begin{Bmatrix} r & u \\ & m \end{Bmatrix} + X_{rmt}\begin{Bmatrix} s & u \\ & m \end{Bmatrix} + X_{rsm}\begin{Bmatrix} t & u \\ & m \end{Bmatrix}\right]$$

and so on. These derivatives of higher order may also be written neatly by using matrical notation, but we shall carry that method no further.

A particular case of interest is the successive covariant derivatives of a function F. The first is merely the set $X_r = \partial F/\partial x_r$ as shown above (§ 14); the second is

$$X_{rs} = \frac{\partial^2 F}{\partial x_r \partial x_s} - \Sigma_m \frac{\partial F}{\partial x_m}\begin{Bmatrix} r & s \\ & m \end{Bmatrix}.$$

In this particular case the system is symmetric, $X_{rs} = X_{sr}$, because the Christoffel symbols are, as is known, symmetric. Moreover [21] if the covariant derivatives X_{rs} of a system X_r form a symmetric system $X_{rs} = X_{sr}$, then the elements X_r of the system, must be the partial derivatives of the same function F.

18. Contravariant differentiation. If $X^{(r)}$ is a contravariant system of order 1, we should call a contravariant set $X^{(rs)}$ of order 2, which contains the derivatives $\partial X^{(r)}/\partial x_s$ and the coefficients a_{rs} and their derivatives, the first contravariant derived set. We may obtain this set by considering the dual $X^{(rs)}$ of the first covariant derivative X_{rs} of the set X_r, dual to the given set $X^{(r)}$. Thus,

$$X^{(uv)} = \Sigma_{rs} a^{(ru)} a^{(sv)} X_{rs} = \Sigma_{rs} a^{(ru)} a^{(sv)} \left[\frac{\partial X_r}{\partial x_s} - \Sigma_m X_m \begin{Bmatrix} r & s \\ m \end{Bmatrix} \right]$$

$$= \Sigma_{sr} a^{(sv)} \left[\frac{\partial a^{(ru)} X_r}{\partial x_s} - X_r \frac{\partial a^{(ru)}}{\partial x_s} - a^{(ru)} \Sigma_m X_m \begin{Bmatrix} r & s \\ m \end{Bmatrix} \right]$$

$$= \Sigma_s a^{(sv)} \left[\frac{\partial X^{(u)}}{\partial x_s} - \Sigma_{rt} X^{(t)} a_{rt} \frac{\partial a^{(ru)}}{\partial x_s} - \Sigma_{rmt} a^{(ru)} X^{(t)} a_{mt} \begin{Bmatrix} r & s \\ m \end{Bmatrix} \right]$$

Now as $\Sigma_r a_{rt} a^{(ru)} = \epsilon_{ut}$, we have

$$\Sigma_r a_{rt} \frac{\partial a^{(ru)}}{\partial x_s} = -\Sigma_r a^{(ru)} \frac{\partial a_{rt}}{\partial x_s}.$$

Hence

$$X^{(uv)} = \Sigma_s a^{(sv)} \left[\frac{\partial X^{(u)}}{\partial x_s} + \Sigma_{rt} X^{(t)} a^{(ru)} \frac{\partial a_{rt}}{\partial x_s} - \Sigma_{rt} a^{(ru)} X^{(t)} \begin{bmatrix} r & s \\ t \end{bmatrix} \right].$$

But, by (32),

$$\frac{\partial a_{rt}}{\partial x_s} - \begin{bmatrix} r & s \\ t \end{bmatrix} = \begin{bmatrix} t & s \\ r \end{bmatrix}. \qquad (32')$$

Hence

$$X^{(uv)} = \Sigma_s a^{(sv)} \left[\frac{\partial X^{(u)}}{\partial x_s} + \Sigma_t X^{(t)} \begin{Bmatrix} t & s \\ u \end{Bmatrix} \right]. \qquad (37)$$

[21] See Ricci, *Lezioni*, p. 70.

For a contravariant system of higher order the process is similar and the result is as follows:

$$X^{(uvw)} = \Sigma_s a^{(sw)} \left[\frac{\partial X^{(uv)}}{\partial x_s} + \Sigma_t \left(X^{(ut)} \left\{ \begin{matrix} t\ s \\ v \end{matrix} \right\} + X^{(tv)} \left\{ \begin{matrix} t\ s \\ u \end{matrix} \right\} \right) \right], \tag{37'}$$

and similarly in general.

The partial derivatives of a contravariant set may then be obtained by solution. For,

$$\Sigma_v X^{(uv)} a_{rv} = \frac{\partial X^{(u)}}{\partial x_r} + \Sigma_t X^{(t)} \left\{ \begin{matrix} t\ r \\ v \end{matrix} \right\}, \tag{38}$$

$$\Sigma_w X^{(uvw)} a_{rw} = \frac{\partial X^{(uv)}}{\partial x_r} + \Sigma_t \left(X^{(ut)} \left\{ \begin{matrix} t\ r \\ v \end{matrix} \right\} + X^{(tv)} \left\{ \begin{matrix} t\ r \\ u \end{matrix} \right\} \right). \tag{38'}$$

19. Properties of covariant differentiation. If we apply (36) to the set a_{rs} of the coefficients of the quadratic differential form, we find

$$\begin{aligned} a_{rst} &= \frac{\partial a_{rs}}{\partial x_t} - \Sigma_m \left[a_{ms} \left\{ \begin{matrix} r\ t \\ m \end{matrix} \right\} + a_{rm} \left\{ \begin{matrix} s\ t \\ m \end{matrix} \right\} \right] \\ &= \frac{\partial a_{rs}}{\partial x_t} - \left[\begin{matrix} r\ t \\ s \end{matrix} \right] - \left[\begin{matrix} s\ t \\ r \end{matrix} \right] = 0, \end{aligned}$$

as follows from (32) and (32'). Hence the first covariant derived set of a_{rs} vanishes identically. The same may be proved of the first contravariant derived set of $a^{(rs)}$; but as the set $a^{(rs)}$ is the dual of the set a_{rs}, no formal proof is necessary.

The covariant derivatives of a product of covariant factors follows the rule of ordinary differentiation. For example,

$$(X_r X_s)_t = X_{rt} X_s + X_r X_{st}, \tag{39}$$

since

$$\begin{aligned} (X_r X_s)_t &= \frac{\partial X_r X_s}{\partial x_t} - \Sigma_m \left[X_m X_s \left\{ \begin{matrix} r\ t \\ m \end{matrix} \right\} + X_r X_m \left\{ \begin{matrix} s\ t \\ m \end{matrix} \right\} \right] \\ &= \left[\frac{\partial X_r}{\partial x_t} - \Sigma_m X_m \left\{ \begin{matrix} r\ t \\ m \end{matrix} \right\} \right] X_s + \left[\frac{\partial X_s}{\partial x} - \Sigma_m X_m \left\{ \begin{matrix} s\ t \\ m \end{matrix} \right\} \right] X_r . \end{aligned}$$

The covariant derivative of the covariant system formed by the composition of a contravariant system of order m and a covariant system of order $m + p$ may be written

$$Z_{i_1 i_2 \ldots i_p t} = \Sigma_{j_1 j_2 \ldots j_m} X_{i_1 i_2 \ldots i_p j_1 j_2 \ldots j_p t} Y^{(j_1 j_2 \ldots j_m)} + \Sigma_{j_1 j_2 \ldots j_m s} X_{i_1 i_2 \ldots i_p j_1 j_2 \ldots j_m} Y^{(j_1 j_2 \ldots j_m s)} a_{st} \qquad (40)$$

There is a dual proposition for the contravariant derivative of a contravariant system formed by composition of a covariant system of order p and a contravariant system of order $m + p$.

A special case of importance is the differentiation of the invariant which arises from the composition when the orders of the covariant and contravariant systems are equal. We have, from (40),

$$I_t = \Sigma_{j_1 j_2 \ldots j_m} X_{j_1 j_2 \ldots j_m t} Y^{(j_1 j_2 \ldots j_m)} + \Sigma_{j_1 j_2 \ldots j_m s} X_{j_1 j_2 \ldots j_m} Y^{(j_1 j_2 \ldots j_m s)} a_{st} .$$

If we write for $Y^{(j_1 j_2 \ldots j_m s)}$ its value

$$Y^{(j_1 j_2 \ldots j_m s)} = \Sigma_{i_1 i_2 \ldots i_m r} a^{(i_1 j_1)} a^{(i_2 j_2)} \cdots a^{(i_m j_m)} a^{(rs)} Y_{i_1 i_2 \ldots i_m r} ,$$

we may sum over the i's combining the a's with the X's; then, with proper change of indices,

$$I_t = \Sigma_{j_1 j_2 \ldots j_m} [X_{j_1 j_2 \ldots j_m t} Y^{(j_1 j_2 \ldots j_m)} + X^{(j_1 j_2 \ldots j_m)} Y_{j_1 j_2 \ldots j_m t}] . \qquad (41)$$

20. Relative covariant differentiation.— Covariant differentiation is a process which derives from a covariant set of order m another covariant set of order $m + 1$ containing the derivatives of the elements of the first set and certain derivatives of the coefficients of the quadratic form, namely the Christoffel symbols. We may obtain a covariant set of order $m + 1$ from one of order m in other ways, without the use of Christoffel symbols but with the aid of the functions which define an n-tuple and its reciprocal.

Let us express X_r in terms of the λ's as a basis (§ 12).

$$X_r = \Sigma_i c_{i \cdot i} \lambda'_r , \qquad c_i = \Sigma_{t \cdot i} \lambda^{(t)} X_t .$$

Now differentiate with respect to x_s. Then.

$$\frac{\partial X_r}{\partial x_s} = \Sigma_i c_i \frac{\partial_i \lambda'_r}{\partial x_s} + \Sigma_i \frac{\partial c_i}{\partial x_s} {}_i\lambda'_r . \tag{42}$$

We next observe that ${}_i\lambda'_r$ is covariant and that

$$\frac{\partial c_i}{\partial x_s} = \Sigma_t \frac{\partial c_i}{\partial y_t} \frac{\partial y_t}{\partial x_s} .$$

As c_i is an invariant, $\partial c_i / \partial y_t$ is the expression in the new variables corresponding to $\partial c_i / \partial x_s$. If we introduce the new λ''s, we have

$$\Sigma_i \frac{\partial c_i}{\partial x_s} {}_i\lambda'_r = \Sigma_{itu} \left(\frac{\partial c_i}{\partial y_t}\right) ({}_i\lambda'_u) \frac{\partial y_t}{\partial x_s} \frac{\partial y_u}{\partial x_r} .$$

Hence the set of terms $\Sigma_i \cdot {}_i\lambda'_r \partial c_i / \partial x_s$ is covariant of order 2. Now, replacing in (42) the invariant c_i by $\Sigma_t \cdot {}_i\lambda^{(t)} X_t$ and transposing, we have as a covariant set of order 2,

$$X_{rs} = \frac{\partial X_r}{\partial x_s} - \Sigma_t X_t \Sigma_i \cdot {}_i\lambda^{(t)} \frac{\partial_i \lambda'_r}{\partial x_s} . \tag{43}$$

(We may verify directly that X_{rs} is a covariant set of order 2 by transforming it.)

If we had a set of order 2 expressed in terms of the basis, we find

$$X_{rs} = \Sigma_{ij} c_{ij} \cdot {}_i\lambda'_r \cdot {}_j\lambda'_s , \qquad \text{with } c_{ij} = \Sigma_{pq} X_{pq} \cdot {}_i\lambda^{(p)} {}_j\lambda^{(q)} .$$

Differentiate and transpose,

$$\frac{\partial X_{rs}}{\partial x_t} - \Sigma_{ij} c_{ij} \frac{\partial_i \lambda'_r}{\partial x_t} {}_j\lambda'_s - \Sigma_{ij} c_{ij} \cdot {}_i\lambda'_r \frac{\partial_j \lambda'_s}{\partial x_t} = \Sigma_{ij} \frac{\partial c_{ij}}{\partial x_t} {}_i\lambda'_r \cdot {}_j\lambda'_s .$$

The right hand member forms (for all different values of r, s, t,) a covariant system of order 3; so also must the left hand member. If now we replace c_{ij} by its value and if we note that $\Sigma_j \cdot {}_j\lambda^{(q)} {}_j\lambda'_s = \epsilon_{sq}$ by (21′), we see that

$$X_{rst} = \frac{\partial X_{rs}}{\partial x_t} - \Sigma_p X_{ps} \Sigma_i \cdot {}_i\lambda^{(p)} \frac{\partial ({}_i\lambda'_r)}{\partial x_t} - \Sigma_p X_{rp} \Sigma_i \cdot {}_i\lambda^{(p)} \frac{\partial ({}_i\lambda'_s)}{\partial x_t} \tag{43'}$$

is a covariant system of order 3. And in like manner we could form from a system of order m a covariant system of order $m + 1$.

CHAPTER II. THE GENERAL THEORY OF SURFACES.

21. Normalization of element of arc. In ordinary surface theory the second fundamental form may be derived [22] by considering a change of variable from the given or first fundamental form,

$$\varphi = \Sigma a_{rs} dx_r dx_s \text{ to } (\varphi) = \Sigma dy_k^2,$$

where r, s have the range 1, 2, and k the range 1, 2, 3. We shall refer to Ricci [23] for this development and proceed to the case in which we are interested, namely, in which the surface lies not in 3 but in $n > 3$ dimensions. The proof for this case is similar to Ricci's. We shall treat first the simplest assumption, namely, that $n = 4$, and shall mention the generalization to $n > 4$ for the most part without proof.

To simplify notations we shall use a small amount of vector analysis. A set of values of the variables y_i may be written simply as $\mathbf{y}$. A sum of the form $\Sigma_k y_k z_k$ is then the *scalar product* $\mathbf{y} \cdot \mathbf{z}$. The use of vector analysis is possible and entirely appropriate when operating as now in a Euclidean space of n dimensions. If any question as to the legitimacy of the application of Ricci's rules for the absolute calculus arises we may revert at once to the ordinary form of analysis without vectors by taking components (supposed to be along fixed orthogonal directions) of vector equations and by replacing scalar products by sums.

We have, then,

$$\Sigma_k dy_k^2 = d\mathbf{y} \cdot d\mathbf{y} = \Sigma a_{rs} dx_r dx_s, \qquad k = 1, \ldots, 4;\ r, s = 1, 2,$$

by virtue of some transformation

$$y_k = y_k(x_1, x_2) \qquad \text{or} \qquad \mathbf{y} = \mathbf{y}(x_1, x_2).$$

Now if $\mathbf{y}_r$ denote the partial derivative of $\mathbf{y}$ by x_r,

$$d\mathbf{y} \cdot d\mathbf{y} = \Sigma_{rs} \mathbf{y}_r \cdot \mathbf{y}_s dx_r dx_s, \tag{44}$$

22 It is not ordinarily derived in this way.
23 *Lezioni*, Part II, Chap. 1.

and hence

$$a_{rs} = \mathbf{y}_r \cdot \mathbf{y}_s. \tag{44'}$$

Differentiate covariantly by the rule for a composed system (§ 19)

$$a_{rst} = 0 = \mathbf{y}_{rt} \cdot \mathbf{y}_s + \mathbf{y}_r \cdot \mathbf{y}_{st}.$$

As this relation holds for any r, s, t, we have also

$$0 = \mathbf{y}_{tr} \cdot \mathbf{y}_s + \mathbf{y}_t \cdot \mathbf{y}_{rs}, \qquad 0 = \mathbf{y}_{rs} \cdot \mathbf{y}_t + \mathbf{y}_r \cdot \mathbf{y}_{ts}.$$

As $\mathbf{y}$ is a function of x_1, x_2, the second covariant derivative is commutative like an ordinary derivative (§ 19), and by addition and subtraction among the three equations we have

$$\mathbf{y}_t \cdot \mathbf{y}_{rs} = 0 \qquad \text{or} \qquad \mathbf{y}_r \cdot \mathbf{y}_{st} = 0, \tag{45}$$

for all values of r, s, t.

22. Normal vectors. Equations (45) mean that the second covariant derivatives $\mathbf{y}_{st}$ are perpendicular to the first derivatives $\mathbf{y}_r$. As $\mathbf{y}_r$ lies in the tangent plane and as $\mathbf{y}_{st}$ is perpendicular to $\mathbf{y}_r$ for $r = 1, 2$, we infer that the vectors $\mathbf{y}_{st}$ lie in the normal plane [24] to the surface.[25] If $\mathbf{z}$ and $\mathbf{w}$ are any two unit vectors in the normal plane we may write

$$\mathbf{y}_{rs} = b_{rs}\mathbf{z} + c_{rs}\mathbf{w} \tag{46}$$

with

$$\mathbf{z} \cdot \mathbf{z} = \mathbf{w} \cdot \mathbf{w} = 1, \qquad \mathbf{z} \cdot \mathbf{w} = 0, \tag{47}$$

$$\mathbf{z} \cdot \mathbf{y}_r = \mathbf{w} \cdot \mathbf{y}_r = 0, \tag{47'}$$

$$b_{rs} = \mathbf{z} \cdot \mathbf{y}_{rs}, \qquad c_{rs} = \mathbf{w} \cdot \mathbf{y}_{rs}. \tag{47''}$$

Here $\mathbf{z}$, $\mathbf{w}$ are particular unit vectors in the normal plane and consequently are invariant of the coordinate system, x_1, x_2; they are,

24 By the normal plane we mean the plane which is completely perpendicular to the tangent plane, that is, such that any line in one is perpendicular to every line in the other. These planes intersect in only one point.

25 One great advantage of the covariant derivative is therefore brought to light; for the ordinary second derivative of $\mathbf{y}$ would not lie in the normal plane.

however, functions of x_1, x_2, namely, invariant functions. The set of quantities b_{rs}, c_{rs} are therefore covariant. As $\mathbf{y}_{rs} = \mathbf{y}_{sr}$ we see that b_{rs} and c_{rs} are also symmetrical sets.

We may differentiate (47′) to find the derivatives of $\mathbf{z}$ and $\mathbf{w}$. Then

$$\mathbf{y}_{rs}\cdot\mathbf{z} + \mathbf{y}_r\cdot\mathbf{z}_s = 0, \qquad \mathbf{y}_{rs}\cdot\mathbf{w} + \mathbf{y}_r\cdot\mathbf{w}_s = 0.$$

Hence

$$\mathbf{y}_r\cdot\mathbf{z}_s = -\,b_{rs}, \qquad \mathbf{y}_r\cdot\mathbf{w}_s = -\,c_{rs}.$$

Also, from (47),

$$\mathbf{z}\cdot\mathbf{z}_s = 0, \qquad \mathbf{w}\cdot\mathbf{w}_s = 0, \qquad \mathbf{z}\cdot\mathbf{w}_s + \mathbf{w}\cdot\mathbf{z}_s = 0.$$

Let

$$\mathbf{z}\cdot\mathbf{w}_s = +\,\nu_s, \qquad \mathbf{w}\cdot\mathbf{z}_s = -\,\nu_s. \tag{48}$$

We have then four equations (since $r = 1, 2$) to solve for $\mathbf{z}_s$; one of the equations shows that $\mathbf{z}_s$ is perpendicular to $\mathbf{z}$ and the other three give the components of $\mathbf{z}_s$ along the tangent plane and along $\mathbf{w}$. Now

$$\mathbf{y}^{(p)}\cdot\mathbf{y}_r = \Sigma_h y_h{}^{(p)} y_{h|r} = \Sigma_{ht} y_{h|t} a^{(pt)} y_{h|r} = \Sigma_t a^{(pt)} a_{rt} = \epsilon_{rp}.$$

The solution for $\mathbf{z}_s$ may then be written by inspection as

$$\mathbf{z}_s = -\,\Sigma_p b_{ps}\mathbf{y}^{(p)} - \nu_s\mathbf{w}, \tag{49}$$

and checked; in like manner,

$$\mathbf{w}_s = -\,\Sigma_p c_{ps}\mathbf{y}^{(p)} + \nu_s\mathbf{z}. \tag{49'}$$

23. Gauss-Codazzi relations. The third derivatives of $\mathbf{y}$ may next be found by differentiating (covariantly) the expressions (46).

$$\mathbf{y}_{rst} = b_{rst}\mathbf{z} + c_{rst}\mathbf{w} + b_{rs}\mathbf{z}_t + c_{rs}\mathbf{w}_t,$$

or

$$\mathbf{y}_{rst} = \mathbf{z}[b_{rst} + c_{rs}\nu_t] + \mathbf{w}[c_{rst} - b_{rs}\nu_t] - \Sigma_p[b_{pt}b_{rs} + c_{pt}c_{rs}]\mathbf{y}^{(p)}. \tag{50}$$

Now by (36) the general form for a third derivative is

$$X_{rst} = \frac{\partial X_{rs}}{\partial x_t} - \Sigma_m \left[X_{ms} \left\{ \begin{matrix} r\ \ t \\ m \end{matrix} \right\} + X_{rm} \left\{ \begin{matrix} s\ \ t \\ m \end{matrix} \right\} \right]$$

$$= \frac{\partial^2 X_r}{\partial x_s \partial x_t} - \Sigma_m \left[X_m \frac{\partial}{\partial x_t} \left\{ \begin{matrix} r\ \ s \\ m \end{matrix} \right\} + \left\{ \begin{matrix} r\ \ s \\ m \end{matrix} \right\} \frac{\partial X_m}{\partial x_t} \right] - \Sigma_m \left[X_{ms} \left\{ \begin{matrix} r\ \ t \\ m \end{matrix} \right\} + X_{rm} \left\{ \begin{matrix} s\ \ t \\ m \end{matrix} \right\} \right]$$

and

$$\frac{\partial X_m}{\partial x_t} = X_{mt} + \Sigma_p X_p \left\{ \begin{matrix} m\ \ t \\ p \end{matrix} \right\}.$$

Hence

$$X_{rst} = \frac{\partial^2 X_r}{\partial x_s \partial x_t} - \Sigma_m X_m \frac{\partial}{\partial x_t} \left\{ \begin{matrix} r\ \ s \\ m \end{matrix} \right\} - \Sigma_m \left\{ \begin{matrix} r\ \ s \\ m \end{matrix} \right\} X_{mt}$$

$$- \Sigma_{mp} X_p \left\{ \begin{matrix} r\ \ s \\ m \end{matrix} \right\} \left\{ \begin{matrix} m\ \ t \\ p \end{matrix} \right\} - \Sigma_m \left[X_{ms} \left\{ \begin{matrix} r\ \ t \\ m \end{matrix} \right\} + X_{rm} \left\{ \begin{matrix} s\ \ t \\ m \end{matrix} \right\} \right].$$

and

$$X_{rst} - X_{rts} = - \Sigma_m X_m \left[\frac{\partial}{\partial x_t} \left\{ \begin{matrix} r\ \ s \\ m \end{matrix} \right\} - \frac{\partial}{\partial x_s} \left\{ \begin{matrix} r\ \ t \\ m \end{matrix} \right\} + \Sigma_p \left(\left\{ \begin{matrix} r\ \ s \\ p \end{matrix} \right\} \left\{ \begin{matrix} p\ \ t \\ m \end{matrix} \right\} - \left\{ \begin{matrix} r\ \ t \\ p \end{matrix} \right\} \left\{ \begin{matrix} p\ \ s \\ m \end{matrix} \right\} \right) \right]$$

$$- \Sigma_m \left[\left\{ \begin{matrix} r\ \ s \\ m \end{matrix} \right\} X_{mt} - \left\{ \begin{matrix} r\ \ t \\ m \end{matrix} \right\} X_{ms} + \left\{ \begin{matrix} r\ \ t \\ m \end{matrix} \right\} X_{ms} - \left\{ \begin{matrix} r\ \ s \\ m \end{matrix} \right\} X_{mt} + X_{rm} \left\{ \begin{matrix} s\ \ t \\ m \end{matrix} \right\} - X_{rm} \left\{ \begin{matrix} s\ \ t \\ m \end{matrix} \right\} \right]$$

$$= - \Sigma_m X_m \left[\frac{\partial}{\partial x_t} \left\{ \begin{matrix} r\ \ s \\ m \end{matrix} \right\} - \frac{\partial}{\partial x_s} \left\{ \begin{matrix} r\ \ t \\ m \end{matrix} \right\} + \Sigma_p \left(\left\{ \begin{matrix} r\ \ s \\ p \end{matrix} \right\} \left\{ \begin{matrix} p\ \ t \\ m \end{matrix} \right\} - \left\{ \begin{matrix} r\ \ t \\ p \end{matrix} \right\} \left\{ \begin{matrix} p\ \ s \\ m \end{matrix} \right\} \right) \right].$$

Thus the difference of the two third derivatives of a function is expressible in terms of the first derivatives X_m and a combination of the derivatives of the Christoffel symbols with the symbols themselves. This combination is the Riemann symbol[26] $\{rm, st\}$ of the second kind and hence

$$X_{rst} - X_{rts} = -\Sigma_m X_m \{rm, st\} = -\Sigma_u X^{(u)}(ru, st), \tag{51}$$

where
$$(ru, st) = \Sigma_m a_{mu} \{rm, st\} \tag{51'}$$

is a Riemann symbol of the first kind. As (ru, st) and (ur, st) differ only in sign, we have

$$X_{rst} - X_{rts} = \Sigma_u X^{(u)}(ur, st). \tag{51''}$$

From (50) we may obtain $\mathbf{y}_{rst} - \mathbf{y}_{rts}$ and identify with

$$\mathbf{y}_{rst} - \mathbf{y}_{rts} = \Sigma_u \mathbf{y}^{(u)}(ur, st). \tag{52}$$

As the vectors $\mathbf{y}^{(u)}$ are tangential, the components of $\mathbf{z}$ and $\mathbf{w}$ vanish in this direction. Hence we obtain the equations,

$$b_{rst} - b_{rts} = c_{rt}\nu_s - c_{rs}\nu_t. \tag{53}$$

$$c_{rst} - c_{rts} = -b_{rs}\nu_t - b_{rt}\nu_s. \tag{53'}$$

$$(pr, st) = [(b_{ps}b_{rt} - b_{pt}b_{rs}) + (c_{ps}c_{rt} - c_{pt}c_{rs})]. \tag{53''}$$

24. Extension to $n > 4$. Thus far the four dimensional case has been treated. The generalization is simple. Instead of two independent normals $\mathbf{z}$, $\mathbf{w}$, we have $n-2$ normals $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{n-2}$ and may write

$$\mathbf{y}_{rs} = {}_1b_{rs}\mathbf{z}_1 + {}_2b_{rs}\mathbf{z}_2 + \ldots + {}_{n-2}b_{rs}\mathbf{z}_{n-2}, \tag{54}$$

$$\mathbf{z}_i \cdot \mathbf{z}_j = \epsilon_{ij}, \quad \mathbf{z}_i \cdot \mathbf{y}_s = 0, \quad i = 1, 2, \ldots, n-2. \tag{55}$$

If we differentiate, we have

$$\mathbf{z}_{i|s} \cdot \mathbf{z}_j + \mathbf{z}_i \cdot \mathbf{z}_{j|s} = 0, \quad \mathbf{z}_{i|r} \cdot \mathbf{y}_s + \mathbf{z}_i \cdot \mathbf{y}_{rs} = 0,$$

26 Pascal, *Repertorio* (Italian), Vol. II, p. 850, except for a typographical error.

If $i \neq j$, we set

$$\mathbf{z}_i \cdot \mathbf{z}_{j|s} = \nu_{ij|s}, \quad \mathbf{z}_j \cdot \mathbf{z}_{i|s} = -\nu_{ij|s} = \nu_{ji|s}, \tag{56}$$

and

$$\mathbf{z}_{i|r} \cdot \mathbf{y}_s = -\,{}_ib_{rs}. \tag{56'}$$

We can then obtain by the same process as before,

$${}_ib_{rst} - {}_ib_{rts} = \Sigma_{j=1}^{n-2}\,({}_jb_{rs}\nu_{ji|t} - {}_jb_{rt}\nu_{ji|s}), \tag{57}$$

$$(pr, st) = \Sigma_{i=1}^{n-2}\,({}_ib_{ps}\cdot{}_ib_{rt} - {}_ib_{pt}\cdot{}_ib_{rs}). \tag{57'}$$

Moreover we may obtain by a somewhat detailed analysis in the case $n = 4, 5, \ldots$ a relation involving the second derivatives of ν as

$$\nu_{rs} - \nu_{sr} = \Sigma_{pq}\,(b_{pr}c_{qs} - b_{ps}c_{qr})\,a^{(pq)}, \quad n = 4, \tag{58}$$

$$\nu_{ji|rs} - \nu_{ji|sr} + \Sigma_{l=1}^{n-2}\,(\nu_{lj|r}\,\nu_{li|s} - \nu_{lj|s}\,\nu_{li|r})$$

$$= \Sigma_{pq}a^{(pq)}\,({}_ib_{pr}\cdot{}_jb_{qs} - {}_ib_{ps}\cdot{}_jb_{qr})]. \tag{58'}$$

In the case of a binary (first) fundamental form $\varphi = \Sigma a_{rs}dx_rdx_s$, the Riemann symbol (pr, st) reduces to a single one, namely $(12, 12)$, and we may write

$$(12, 12) = aG, \tag{59}$$

where G is an invariant, $(G) = G$, called the Gaussian invariant or *Gaussian curvature*. If $n = 4$ equation (53″) may be written

$$|\,b\,| + |\,c\,| = aG, \tag{59'}$$

and in higher dimensions we have, from (57′),

$$\Sigma_i\,|\,{}_ib\,| = aG, \tag{59''}$$

where $|\,b\,|$, $|\,c\,|$, $|\,{}_ib\,|$ are the determinants formed of the terms b_{rs}, c_{rs}, ${}_ib_{rs}$. In case $n = 3$ we have simply $|\,b\,| = aG$.

25. The Vector Second Form. In three dimensions we construct a form,

$$\psi = \Sigma b_{rs}dx_rdx_s,$$

from the symmetric system b_{rs} and call it the second fundamental form of the surface, defined by the first form φ as one of a class of applicables, and thus we have the surface defined by φ and ψ as a rigid surface. In higher dimensions we construct $n-2$ forms $\psi_1, \psi_2, \ldots, \psi_{n-2}$ (two, when $n = 4$) from the $n-2$ symmetric systems ${}_ib_{rs}$ and this set of $n-2$ forms are *the second fundamental forms*. The different forms are not, however, entirely determined because with a different choice of the unit vectors $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{n-2}$ in the normal $(n-2)$-space, there is a change in the quantities ${}_ib_{rs}$. The set of forms ψ_i taken with φ and the generalized Gauss-Codazzi relations (57), (57′), (58), (58′) will determine the surface as a rigid surface in n-dimensions.[27] We shall not, however, enter into a proof of this proposition which is adequately treated by Ricci and not important for our work.

It has been stated that the systems ${}_ib_{rs}$ are not entirely determined. The relations between different systems may be illustrated in the case $n = 4$. We had $\mathbf{y}_{rs} = b_{rs}\mathbf{z} + c_{rs}\mathbf{w}$, that is, b_{rs} and c_{rs} are respectively the components of $\mathbf{y}_{rs}$ along $\mathbf{z}$ and along $\mathbf{w}$. If a new choice $\mathbf{z}'$, $\mathbf{w}'$ were made, the quantities b'_{rs}, c'_{rs} would be the components of $\mathbf{y}_{rs}$ along $\mathbf{z}'$, $\mathbf{w}'$. Hence the relations b'_{rs}, c'_{rs} and b_{rs}, c_{rs} are those which express a rotation, namely,

$$b_{rs} = b'_{rs}\cos\theta - c'_{rs}\sin\theta, \quad c_{rs} = b'_{rs}\sin\theta + c'_{rs}\cos\theta.$$

In general if $n > 4$, the relations between ${}_ib_{rs}$ and ${}_ib'_{rs}$ must be those which determine an orthogonal transformation in the normal $(n-2)$-space, since ${}_ib_{rs}$ and ${}_ib'_{rs}$ are merely the components of $\mathbf{y}_{rs}$ along two different systems of orthogonal lines in that space. This amount and only this amount of indetermination is involved in our set of second fundamental forms ψ_i.

27 The generalization of the Gauss-Codazzi equations to hypersurfaces (for which the element of arc is a quadratic form of class 1) has been obtained by a number of authors, including Ricci, and do not contain the ν's which by Ricci's development (*Lezioni*, Introduction, Chap. 4) are necessary in case the class of the surface is greater than one. Levi (loc. cit., note 2) develops the theory of surfaces in a very different way. For him the element of arc is apparently not a particularly fundamental form but merely one of a set of fundamental forms. That is to say where we, following Ricci, have a first fundamental form (which is scalar) and a second fundamental form (60) which is vectorial, both quadratic, Levi has an infinite set of $(\mu + \nu)$-linear forms $F_{\mu\nu}$ $(\mu, \nu = 1, 2, \ldots)$ of which the first, F_{11}, is ds^2. He shows that the problem of finding the absolute invariants reduces to that of finding the simultaneous invariants of the forms $F_{\mu\nu}$ and he finds five special invariants Δ_i $(i = 1, \ldots, 5)$ which form a complete system of independent invariants. Our analysis leads us very naturally to five invariants which are equivalent to Levi's (see note 39).

Instead of carrying $n-2$ second fundamental forms ψ_i we shall combine them into *a single vector second fundamental form*

$$\Psi = \mathbf{z}_1\psi_1 + \mathbf{z}_2\psi_2 + \ldots\ldots + \mathbf{z}_{n-2}\psi_{n-2} = \Sigma \mathbf{y}_{rs} dx_r dx_s \qquad (60)$$

in the normal $(n-2)$*-space.* If the vector form is regarded as given, the surface may be regarded as not fixed relative to arbitrary axes in space; only the shape of the surface is determined.

26. Canonical orthogonal curve systems.[28] We have defined a set of curves on a surface by the differential equations obtained by equating the ratios $dx_r : \lambda^{(r)}$(§13; here $r = 1, 2$). The quantities $\lambda^{(r)}$ are the contravariant system defining the curves; the dual system λ_r is a covariant system which may also be regarded as defining the curves. We have defined perpendicularity and hence orthogonal systems of curves. If we give the definition

$$\lambda^{(r)} = \frac{dx_r}{ds} \qquad (61)$$

we have a special system $\lambda^{(r)}$ which satisfies the relation

$$\Sigma_r \lambda_r \lambda^{(r)} = 1, \qquad (61')$$

and we shall here assume this system. The orthogonal curves defined by $\overline{\lambda}^{(r)}$ or $\overline{\lambda}_r$ will satisfy the relation

$$\Sigma_r \lambda^{(r)} \overline{\lambda}_r = \Sigma_r \lambda_r \overline{\lambda}^{(r)} = 0. \qquad (62)$$

If we impose the further condition

$$\Sigma_r \overline{\lambda}_r \overline{\lambda}^{(r)} = 1, \qquad (62')$$

we have a set of relations which will determine $\overline{\lambda}^{(r)}$ or $\overline{\lambda}_r$ except for sign (the arbitrariness of sign corresponds to the two opposite directions along the curve). For from (62) $\overline{\lambda}^{(r)} = (-1)^{r+1}\rho\lambda_{r+1}$, it being understood that all even values of the index are equivalent and all odd values also equivalent. Then from (62′),

$$\Sigma_{rs} \overline{\lambda}^{(r)} \overline{\lambda}^{(s)} a_{rs} = 1 = \Sigma_{rs} \rho^2 (-1)^{r+s} \lambda_{r+1} \lambda_{s+1} a_{rs}.$$

28 Ricci, *Lezioni*, p. 106, and *Atti. R. Ist. Veneto,* (7) **4,** 1–29 (1893).

Now $$a^{(r+1,\,s+1)} = (-1)^{r+s} a_{rs}/a.$$

Hence $$1/\rho^2 = a\Sigma_{rs}\lambda_{r+1}\lambda_{s+1}a^{(r+1,\,s+1)} = a\Sigma_r\lambda_{r+1}\lambda^{(r+1)} = a,$$

and $$\rho = 1/\sqrt{a}.$$

Hence the system $\overline{\lambda}^{(r)}$ is

$$\overline{\lambda}^{(r)} = (-1)^{r+1}\lambda_{r+1}/\sqrt{a}.$$

Further we see easily that

$$\overline{\lambda}_r = (-1)^{r+1}\sqrt{a}\lambda^{(r+1)}. \tag{63}$$

The system $\overline{\lambda}^{(r)}$ or $\overline{\lambda}_r$ is called the canonical orthogonal system for $\lambda^{(r)}$ or λ_r. The repetition of the process of forming the canonical system leads to the negative of the original system (not to the system itself). For

$$\overline{\overline{\lambda}}_r = (-1)^{r+1}\sqrt{a}\,\overline{\lambda}^{(r+1)} = (-1)^3\sqrt{a}\lambda_r/\sqrt{a} = -\lambda_r.$$

If we have a given system $\lambda^{(r)}$ and let φ_s be the covariant system obtained by the composition

$$\varphi_s = \Sigma_r\overline{\lambda}^{(r)}\lambda_{rs}, \tag{64}$$

we have by solution, as may easily be verified,

$$\lambda_{rs} = \overline{\lambda}_r\varphi_s. \tag{64$'$}$$

Also $$\varphi_s = -\Sigma_r\lambda^{(r)}\overline{\lambda}_{rs}, \quad \overline{\lambda}_{rs} = -\lambda_r\varphi_s. \tag{64$''$}$$

Thus by the introduction of φ_s the system λ_{rs} of order two is written as the product $\overline{\lambda}_r\varphi_s$ of two systems of the first order and at the same time $\overline{\lambda}_{rs}$ appears as the product $-\lambda_r\varphi_s$. The system φ_s is called the derived system from the λ's.

27. Expressions of the second forms. If we consider a covariant system b_{rs} we may form the three invariants,

$$\begin{aligned}\alpha &= \Sigma_{rs}\lambda^{(r)}\lambda^{(s)}b_{rs}.\\ \beta &= \Sigma_{rs}\overline{\lambda}^{(r)}\overline{\lambda}^{(s)}b_{rs}.\\ \mu &= \Sigma_{rs}\overline{\lambda}^{(r)}\lambda^{(s)}b_{rs} = \Sigma_{rs}\lambda^{(r)}\overline{\lambda}^{(s)}b_{rs}.\end{aligned} \tag{65}$$

The solution for the b's gives at once,

$$b_{rs} = \alpha\lambda_r\lambda_s + \mu(\bar{\lambda}_r\lambda_s + \lambda_r\bar{\lambda}_s) + \beta\bar{\lambda}_r\bar{\lambda}_s. \tag{65'}$$

The determinant of the b's is then

$$|\, b \,| = a\Sigma_r b_{rs} b^{(rs)} = a(\alpha\beta - \mu^2). \tag{65''}$$

If we are working with several systems ${}_ib_{rs}$ we have for each a set of invariants α_i, β_i, μ_i formed from (65). The second fundamental forms are therefore

$$\psi_i = \Sigma_{rs}[\alpha_i\lambda_r\lambda_s + \mu_i(\lambda_r\bar{\lambda}_s + \bar{\lambda}_r\lambda_s) + \beta_i\bar{\lambda}_r\bar{\lambda}_s]dx_r dx_s. \tag{66}$$

The vector fundamental form is

$$\Psi = \Sigma_{rs}[\boldsymbol{\alpha}\lambda_r\lambda_s + \boldsymbol{\mu}(\lambda_r\bar{\lambda}_s + \bar{\lambda}_r\lambda_s) + \boldsymbol{\beta}\bar{\lambda}_r\bar{\lambda}_s]\, dx_r dx_s \tag{67}$$

where $$\boldsymbol{\alpha} = \Sigma\alpha_i\mathbf{z}_i, \quad \boldsymbol{\mu} = \Sigma\mu_i\mathbf{z}_i, \quad \boldsymbol{\beta} = \Sigma\beta_i\mathbf{z}_i, \tag{67'}$$

i running from 1 to $n-2$. The vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\mu}$ are invariant vectors in the normal $(n-2)$-space. From (65′), (67′) we have immediately,

$$\mathbf{y}_{rs} = \boldsymbol{\alpha}\lambda_r\lambda_s + \boldsymbol{\mu}(\lambda_r\bar{\lambda}_s + \bar{\lambda}_r\lambda_s) + \boldsymbol{\beta}\bar{\lambda}_r\bar{\lambda}_s. \tag{68}$$

Then from (65″) and (59″) we have,

$$G = \Sigma_i(\alpha_i\beta_i - \mu_i^2) = \boldsymbol{\alpha}\cdot\boldsymbol{\beta} - \boldsymbol{\mu}^2. \tag{69}$$

Hence the result: *The Gaussian invariant G is the scalar product of the vector invariants $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ diminished by the square of the vector invariant $\boldsymbol{\mu}$.*

28. Moving rectangular axes. The elements $\mathbf{y}_r$ or $y_{h|r}$, $h = 1, 2, \ldots, n$, are tangent to the surface. If we form

$$\boldsymbol{\xi} = \Sigma_r\lambda^{(r)}\mathbf{y}_r = \Sigma_r\lambda_r\mathbf{y}^{(r)}, \qquad \boldsymbol{\eta} = \Sigma_r\bar{\lambda}^{(r)}\mathbf{y}_r = \Sigma_r\bar{\lambda}_r\mathbf{y}^{(r)}, \tag{70}$$

we have two vectors tangent to the surface. Moreover these are: 1°, unit vectors; 2°, mutually perpendicular; 3°, tangent respectively

to the curves $\lambda^{(r)}$ and to their orthogonal trajectories $\bar{\lambda}^{(r)}$. To prove 1° and 2° we note that

$$\boldsymbol{\xi}\cdot\boldsymbol{\xi} = \Sigma_{rs}\lambda^{(r)}\lambda^{(s)}\mathbf{y}_r\cdot\mathbf{y}_s = \Sigma_{rs}\lambda^{(r)}\lambda^{(s)}a_{rs} = \Sigma_r\lambda^{(r)}\lambda_r = 1,$$

and similar equations hold. For 3° observe that

$$\boldsymbol{\xi} = \Sigma_t \frac{\partial x_t}{\partial s}\frac{\partial \mathbf{y}_r}{\partial x_t} = \frac{\partial \mathbf{y}_r}{\partial s},$$

$d\mathbf{y}_r$ being the differential along the curves $\lambda^{(r)}$.

In case of four dimensions we shall use $\boldsymbol{\zeta}$, $\boldsymbol{\omega}$ (to correspond with $\boldsymbol{\xi}$, $\boldsymbol{\eta}$) in place of $\mathbf{z}$, $\mathbf{w}$ as the unit normal vectors — in higher dimensions $\mathbf{z}_1, \mathbf{z}_2, \ldots \mathbf{z}_{n-2}$. We have therefore such relations as (47) or (55). The systems $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\boldsymbol{\zeta}$, $\boldsymbol{\omega}$ or $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\mathbf{z}_i$, $i = 1, 2, \ldots, n-2$, are therefore systems of moving axes in which $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ move along definite orthogonal trajectories upon the surface.

The rate of change of the unit vectors $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ are, by covariant differentiation of (70),

$$\boldsymbol{\xi}_r = \Sigma_s\lambda^{(s)}\mathbf{y}_{rs} + \Sigma_s\lambda_{rs}\mathbf{y}^{(s)},$$

From (68), (61′), (62),

$$\begin{aligned}
\Sigma_s\lambda^{(s)}\mathbf{y}_{rs}, &= \Sigma_s[\boldsymbol{\alpha}\lambda_r\lambda_s\lambda^{(s)} + \boldsymbol{\mu}(\lambda_r\bar{\lambda}_s + \bar{\lambda}_r\lambda_s)\lambda^{(s)} + \boldsymbol{\beta}\bar{\lambda}_r\bar{\lambda}_s\lambda^{(s)}] \\
&= \boldsymbol{\alpha}\lambda_r + \boldsymbol{\mu}\bar{\lambda}_r = \Sigma_i(a_i\mathbf{z}_i\lambda_r + \mu_i\mathbf{z}_i\bar{\lambda}_r), \\
&\qquad \Sigma_s\lambda_{rs}\mathbf{y}^{(s)} = \Sigma_s\bar{\lambda}_s\varphi_r\mathbf{y}^{(s)} = \varphi_r\boldsymbol{\eta}.
\end{aligned}$$

Hence

$$\begin{aligned}
\boldsymbol{\xi}_r &= \boldsymbol{\alpha}\lambda_r + \boldsymbol{\mu}\bar{\lambda}_r + \boldsymbol{\eta}\varphi_r, \\
\boldsymbol{\eta}_r &= \boldsymbol{\mu}\lambda_r + \boldsymbol{\beta}\bar{\lambda}_r - \boldsymbol{\xi}\varphi_r.
\end{aligned} \tag{71}$$

The rates of change of the normals are found from the relations (55), (56), (56′).

$$\mathbf{z}_i\cdot\mathbf{y}_r = 0, \qquad \mathbf{z}_{is}\cdot\mathbf{y}_r + \mathbf{z}_i\cdot\mathbf{y}_{rs} = 0,$$
$$\mathbf{z}_{is}\cdot\mathbf{y}_r = -\,{}_ib_{rs} \quad \text{and} \quad \mathbf{z}_{is}\cdot\mathbf{z}_j = \nu_{ji|s}.$$

These equations give the components of $\mathbf{z}_{i|s}$ along the surface and along the normals. Hence,

$$\begin{aligned}
\mathbf{z}_{i|r} &= -\,\Sigma_s b_{rs}\mathbf{y}^{(s)} + \Sigma_j\nu_{ji|r}\mathbf{z}_j \\
&= -\,\Sigma_s[a_i\lambda_r\lambda_s + \mu_i(\bar{\lambda}_r\lambda_s + \lambda_r\bar{\lambda}_s) + \beta_i\bar{\lambda}_r\bar{\lambda}_s]\mathbf{y}^{(s)} + \Sigma_j\nu_{ji|r}\mathbf{z}_j
\end{aligned}$$

or $$\mathbf{z}_{i|r} = -\,\boldsymbol{\xi}(a_i\lambda_r + \mu_i\bar{\lambda}_r) - \boldsymbol{\eta}(\mu_i\lambda_r + \beta_i\bar{\lambda}_r) + \Sigma_j\nu_{ji|r}\mathbf{z}_j. \tag{72}$$

If in four dimensions we use $\boldsymbol{\zeta}$, $\boldsymbol{\omega}$, b_{rs}, c_{rs} to avoid subscripts, we have

$$\begin{aligned}\boldsymbol{\zeta}_r &= -\Sigma_s b_{rs}\mathbf{y}^{(s)} + \nu_r\boldsymbol{\omega}\\ &= -\boldsymbol{\xi}(\alpha_1\lambda_r + \mu_1\bar{\lambda}_r) + \boldsymbol{\eta}(\mu_1\lambda_r + \beta_1\bar{\lambda}_r) + \nu_r\boldsymbol{\omega},\\ \boldsymbol{\omega}_r &= -\Sigma_s c_{rs}\mathbf{y}^{(s)} - \nu_r\boldsymbol{\zeta}\\ &= -\boldsymbol{\xi}(\alpha_2\lambda_r + \mu_2\bar{\lambda}_r) + \boldsymbol{\eta}(\mu_2\lambda_r + \beta_2\bar{\lambda}_r) - \nu_r\boldsymbol{\zeta}.\end{aligned} \tag{72'}$$

It is important to observe that the theory of the 2-surface in four or more dimensions is not the same as the theory of the moving axes: for the 2, or $n-2$ normals, to the surface are to a large extent indeterminate so far as the surface itself is concerned. It is the set of quantities ν which render the normal system definite and upon which the rate of change of the normal vectors depends as in the above equations. The theory of the set of moving axes is a step further than the theory of the surface and as far as the surface alone is concerned we may disregard the ν's so long as we do not need to differentiate the normals. In this respect there is the same difference between surface theory and the theory of moving axes (of which two are tangent to the surface) as between the theory of a twisted curve in three dimensions and the theory of moving axes of which only one is tangent to the curve. If the differential theory of a curve is treated from the point of view of the quadratic form (in one variable), the ν which must be introduced in the case of a twisted curve in three dimensions is related to the radius of torsion. In the curve theory the set of axes is rendered definite by assuming that the normal axes are along the principal normal and binormal and if we desire to keep moving axes in our theory of surfaces it will be desirable to specialize the normal axes in some such way as in the case of curves in three dimensions.

29. Tangent plane and normal space. Two elements which have strictly to do with the surface alone are the tangent plane and the normal plane or $(n-2)$-space. Following the notation of Gibbs (for the outer product) we may write the unit tangent plane and its differential as

$$\mathbf{M} = \boldsymbol{\xi}\times\boldsymbol{\eta}, \qquad d\mathbf{M} = d\boldsymbol{\xi}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times d\boldsymbol{\eta}. \tag{73}$$

The unit normal space is,

$$\begin{aligned}\mathbf{N} &= \boldsymbol{\zeta}\times\boldsymbol{\omega} \qquad \text{or} \qquad \mathbf{N} = \mathbf{z}_1\times\mathbf{z}_2\times\ldots\times\mathbf{z}_{n-2},\\ d\mathbf{N} &= d\boldsymbol{\zeta}\times\boldsymbol{\omega} + \boldsymbol{\zeta}\times d\boldsymbol{\omega} \qquad \text{or} \qquad d\mathbf{N} = d\mathbf{z}_1\times\mathbf{z}_2\times\mathbf{z}_3\times\ldots\times\mathbf{z}_{n-2} + \text{etc.}\end{aligned} \tag{73'}$$

In terms of the notation introduced above we have, from (71),

$$d\boldsymbol{\xi} = \Sigma_r \boldsymbol{\xi}_r dx_r = \boldsymbol{\alpha}\Sigma_r \lambda_r dx_r + \boldsymbol{\mu}\Sigma_r \bar{\lambda}_r dx_r + \boldsymbol{\eta}\Sigma_r \varphi_r dx_r\,,$$
$$d\boldsymbol{\eta} = \Sigma_r \boldsymbol{\eta}_r dx_r = \boldsymbol{\mu}\Sigma_r \lambda_r dx_r + \boldsymbol{\beta}\Sigma_r \bar{\lambda}_r dx_r - \boldsymbol{\xi}\Sigma_r \varphi_r dx_r\,.$$

Hence

$$d\mathbf{M} = \boldsymbol{\alpha}\times\boldsymbol{\eta}\Sigma\lambda_r dx_r + \boldsymbol{\mu}\times\boldsymbol{\eta}\Sigma\bar{\lambda}_r dx_r - \boldsymbol{\mu}\times\boldsymbol{\xi}\Sigma\lambda_r dx_r - \boldsymbol{\beta}\times\boldsymbol{\xi}\Sigma\bar{\lambda}_r dx_r\,. \tag{74}$$

Now $\quad d\mathbf{y} = \Sigma\mathbf{y}_r dx_r\,, \qquad \boldsymbol{\xi} = \Sigma\lambda^{(r)}\mathbf{y}_r\,, \qquad \boldsymbol{\eta} = \Sigma\bar{\lambda}^{(r)}\mathbf{y}_r\,.$

The last two equations may be solved by inspection as

$$\mathbf{y}_r = \boldsymbol{\xi}\lambda_r + \boldsymbol{\eta}\bar{\lambda}_r\,,$$
$$d\mathbf{y} = \boldsymbol{\xi}\Sigma\lambda_r dx_r + \boldsymbol{\eta}\Sigma\bar{\lambda}_r dx_r\,. \tag{74'}$$

Hence

$$\begin{aligned} d\mathbf{y}\times d\mathbf{M} &= \boldsymbol{\xi}\times\boldsymbol{\alpha}\times\boldsymbol{\eta}\Sigma\lambda_r dx_r \Sigma\lambda_r dx_r + \boldsymbol{\xi}\times\boldsymbol{\mu}\times\boldsymbol{\eta}\Sigma\lambda_r dx_r \Sigma\bar{\lambda}_r dx_r \\ &\quad - \boldsymbol{\eta}\times\boldsymbol{\mu}\times\boldsymbol{\xi}\Sigma\bar{\lambda}_r dx_r \Sigma\lambda_r dx_r - \boldsymbol{\eta}\times\boldsymbol{\beta}\times\boldsymbol{\xi}\Sigma\bar{\lambda}_r dx_r \Sigma\bar{\lambda}_r dx_r \\ &= -\,\boldsymbol{\xi}\times\boldsymbol{\eta}\times\{\boldsymbol{\alpha}\Sigma_{rs}\lambda_r\lambda_s dx_r dx_s + \boldsymbol{\beta}\Sigma_{rs}\bar{\lambda}_r\bar{\lambda}_s dx_r dx_s \\ &\qquad + \boldsymbol{\mu}\Sigma_{rs}(\lambda_r\bar{\lambda}_s + \bar{\lambda}_r\lambda_s)dx_r dx_s\} \end{aligned}$$

or

$$d\mathbf{y}\times d\mathbf{M} = -\,\mathbf{M}\times\Sigma_{rs}\mathbf{y}_{rs}dx_r dx_s = -\,\mathbf{M}\times\Psi. \tag{75}$$

This expression may be solved for Ψ by multiplying by $\mathbf{M}$. Thus,[29]

$$\mathbf{M}\cdot(d\mathbf{y}\times d\mathbf{M}) = -\,\mathbf{M}\cdot(\mathbf{M}\times\Psi). \tag{75'}$$

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29 We shall use as a definition of the inner product that due to G. N. Lewis (loc. cit., note 15) which has the advantage over the inner product of Grassmann that it is commutative. The interpretation of the inner product of a p-dimensional parallelepiped and a q-dimensional parallelepiped where $q > p$ is a $(q - p)$-dimensional parallelepiped in the q-space perpendicular to the p-space. The rules of operation with inner products have been developed for a non-Euclidean case by Wilson and Lewis (loc. cit., note 15) and the rules for the Euclidean case are not different except for an occasional change of sign. As the product is distributive the rules may all be verified on or derived from products of unit vectors. (For the transformation used in the text at this point see Wilson and Lewis, p. 439). One of the most important rules is that represented by such expansions as,

$$(\mathbf{m}\times\mathbf{n})\cdot(\mathbf{p}\times\mathbf{q}\times\mathbf{r}) = (\mathbf{p}\times\mathbf{q}\times\mathbf{r})\cdot(\mathbf{m}\times\mathbf{n}) = (\mathbf{m}\times\mathbf{n})\cdot(\mathbf{q}\times\mathbf{r})\mathbf{p} + (\mathbf{m}\times\mathbf{n})\cdot(\mathbf{r}\times\mathbf{p})\mathbf{q} + (\mathbf{m}\times\mathbf{n})\cdot(\mathbf{p}\times\mathbf{q})\mathbf{r}.$$

The general rule is to take from the larger factor as many of its factors as there are factors in the smaller factor to form with them a scalar product, taking all

But $$\mathbf{C}\cdot(\mathbf{b}\times\mathbf{A}) = (\mathbf{C}\cdot\mathbf{A})\mathbf{b} + (\mathbf{b}\cdot\mathbf{C})\cdot\mathbf{A},$$

Here, $\mathbf{M}\cdot d\mathbf{M} = 0$ and $\mathbf{M}\cdot\Psi = 0$. Hence

$$(d\mathbf{y}\cdot\mathbf{M})\cdot d\mathbf{M} = -\Psi. \tag{76}$$

The vector $d\mathbf{y}\cdot\mathbf{M}$ is a 1-vector in $\mathbf{M}$ perpendicular to $d\mathbf{y}$ and $(d\mathbf{y}\cdot\mathbf{M})\cdot d\mathbf{M}$ is a 1-vector in $d\mathbf{M}$ perpendicular to $d\mathbf{y}\cdot\mathbf{M}$.

The expressions (75′) or (76) hold of course in three dimensions as the work by which they were obtained is independent of the number of dimensions, greater than two. In ordinary surface theory we have

$$\Psi = \Sigma_{rs}b_{rs}dx_rdx_s = -\Sigma_h dy_h d\zeta_h = -d\mathbf{y}\cdot d\boldsymbol{\zeta},$$

where ζ_h are the direction cosines of the normal $\boldsymbol{\zeta}$. If we multiply this by $\boldsymbol{\zeta}$ to make a vector form we have

$$\Psi = \boldsymbol{\zeta}\Sigma_{rs}b_{rs}dx_rdx_s = -\boldsymbol{\zeta}(d\mathbf{y}\cdot d\boldsymbol{\zeta}) = -d\boldsymbol{\zeta}\cdot(\boldsymbol{\zeta}\times d\mathbf{y}).$$

The form is expressed in terms of the normal and its differential instead of in terms of the tangent plane and its differential. We may make the change by taking complements,[30]

$$[d\boldsymbol{\zeta}\cdot(\boldsymbol{\zeta}\times d\mathbf{y})]^{**} = -[d\boldsymbol{\zeta}\times(\boldsymbol{\zeta}\times d\mathbf{y})^*]^* = -[d\boldsymbol{\zeta}\times(d\mathbf{y}\cdot\mathbf{M})]^* = (d\mathbf{y}\cdot\mathbf{M})\cdot d\mathbf{M}.$$

We have therefore arrived at *a formula* $\Psi = -(d\mathbf{y}\cdot\mathbf{M})\cdot d\mathbf{M}$ *for the (vector) second fundamental form which is the immediate generalization of the formula in three dimensions.*

If we desire to express the second fundamental form in terms of the normal $(n-2)$-space $\mathbf{N}$ instead of in terms of $\mathbf{M}$ we can do so.

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possible combinations and adding with due regard to sign. For the case in which the two factors are of equal order we have

$$(\mathbf{m}\times\mathbf{n})\cdot(\mathbf{p}\times\mathbf{q}) = \begin{vmatrix} \mathbf{m}\cdot\mathbf{p} & \mathbf{n}\cdot\mathbf{p} \\ \mathbf{m}\cdot\mathbf{q} & \mathbf{n}\cdot\mathbf{q} \end{vmatrix}.$$

These rules for obvious reasons are similar to those for regressive or mixed products and the rule quoted at this point in the text is like Müller's theorems (see Whitehead, *Universal Algebra*, p. 192). The complement, denoted by *, which is used below is similar to Grassmann's supplement, except possibly for sign.

30 See Wilson and Lewis (loc. cit., note 15), p. 435.

30. Square of element of surface. Consider $d\mathbf{M}\cdot d\mathbf{M}$ which is numerically equal to $d\mathbf{N}\cdot d\mathbf{N}$.

$$d\mathbf{M}\cdot d\mathbf{M} = [\boldsymbol{\alpha}\times\boldsymbol{\eta}\Sigma\lambda_r dx_r + \boldsymbol{\mu}\times\boldsymbol{\eta}\Sigma\bar\lambda_r dx_r - \boldsymbol{\mu}\times\boldsymbol{\xi}\Sigma\lambda_r dx_r - \boldsymbol{\beta}\times\boldsymbol{\xi}\Sigma\bar\lambda_r dx_r]^2.$$

Now
$$(\boldsymbol{\alpha}\times\boldsymbol{\eta})\cdot(\boldsymbol{\alpha}\times\boldsymbol{\eta}) = \boldsymbol{\alpha}\cdot\boldsymbol{\alpha}\boldsymbol{\eta}\cdot\boldsymbol{\eta} - (\boldsymbol{\alpha}\cdot\boldsymbol{\eta})^2 = \boldsymbol{\alpha}^2,$$

$$(\boldsymbol{\alpha}\times\boldsymbol{\eta})\cdot(\boldsymbol{\mu}\times\boldsymbol{\eta}) = \boldsymbol{\alpha}\cdot\boldsymbol{\mu}, \qquad (\boldsymbol{\alpha}\times\boldsymbol{\eta})\cdot(\boldsymbol{\mu}\times\boldsymbol{\xi}) = 0, \text{ etc.}$$

$$\begin{aligned} d\mathbf{M}\cdot d\mathbf{M} = {} & \boldsymbol{\alpha}^2\Sigma\lambda_r\lambda_s dx_r dx_s + \boldsymbol{\mu}^2\Sigma\bar\lambda_r\bar\lambda_s dx_r dx_s + \boldsymbol{\mu}^2\Sigma\lambda_r\lambda_s dx_r dx_s \\ & + \boldsymbol{\beta}^2\Sigma\bar\lambda_r\bar\lambda_s dx_r dx_s + 2\boldsymbol{\alpha}\cdot\boldsymbol{\mu}\Sigma\lambda_r\bar\lambda_s dx_r dx_s + 2\boldsymbol{\beta}\cdot\boldsymbol{\mu}\Sigma\bar\lambda_r\lambda_s dx_r dx_s. \end{aligned}$$

By (69) we have $\boldsymbol{\mu}^2 = \boldsymbol{\alpha}\cdot\boldsymbol{\beta} - G$. Hence

$$\begin{aligned} d\mathbf{M}\cdot d\mathbf{M} = {} & - G[\Sigma\lambda_r\lambda_s dx_r dx_s + \bar\lambda_r\bar\lambda_s dx_r dx_s] + \boldsymbol{\alpha}\cdot\boldsymbol{\beta}[\Sigma\lambda_r\lambda_s dx_r dx_s \\ & \qquad + \bar\lambda_r\bar\lambda_s dx_r dx_s] \\ & + \boldsymbol{\alpha}\cdot\boldsymbol{\alpha}\Sigma\lambda_r\lambda_s dx_r dx_s + \boldsymbol{\beta}\cdot\boldsymbol{\beta}\Sigma\bar\lambda_r\bar\lambda_s dx_r dx_s \\ & + (\boldsymbol{\alpha}\cdot\boldsymbol{\mu} + \boldsymbol{\beta}\cdot\boldsymbol{\mu})[\Sigma\lambda_r\bar\lambda_s dx_r dx_s + \Sigma\bar\lambda_r\lambda_s dx_r dx_s]. \end{aligned}$$

Now a_{rs} may be expressed in terms of the λ's as b_{rs} was expressed in (65′). Then,

$$a_{rs} = c_1\lambda_r\lambda_s + c_2(\lambda_r\bar\lambda_s + \bar\lambda_r\lambda_s) + c_3\bar\lambda_r\bar\lambda_s.$$

When the invariants c_1, c_2, c_3 are determined by means of (65) we find $c_1 = 1$, $c_2 = 0$, $c_3 = 1$. Hence

$$a_{rs} = \lambda_r\lambda_s + \bar\lambda_r\bar\lambda_s. \tag{77}$$

$$\begin{aligned} d\mathbf{M}\cdot d\mathbf{M} = {} & - G\Sigma a_{rs} dx_r dx_s + (\boldsymbol{\alpha} + \boldsymbol{\beta})\cdot[\boldsymbol{\alpha}\Sigma\lambda_r\lambda_s dx_r dx_s + \boldsymbol{\mu}\Sigma(\bar\lambda_r\bar\lambda_s \\ & \qquad + \bar\lambda_r\lambda_s) dx_r dx_s + \boldsymbol{\beta}\Sigma\bar\lambda_r\bar\lambda_s dx_r dx_s], \end{aligned}$$

or

$$d\mathbf{M}\cdot d\mathbf{M} = - G\varphi + (\boldsymbol{\alpha} + \boldsymbol{\beta})\cdot\Psi. \tag{78}$$

Hence: *The square of the differential of the tangent plane is equal to the scalar product of the vector invariant* $\boldsymbol{\alpha} + \boldsymbol{\beta}$ *and the vector second fundamental form* Ψ *less the product of the Gaussian invariant* G *and the first fundamental form* φ.

This relation holds also in three dimensions: but in this case $\boldsymbol{\alpha} + \boldsymbol{\beta}$ and Ψ are generally regarded as scalar quantities, $d\mathbf{M}\cdot d\mathbf{M}$ is replaced by the square of the differential of the normal,— and, furthermore,

this quantity is interpreted as the differential of arc of the Gaussian spherical representation of the surface. No spherical representation of the same simple sort as obtained in three dimensions exists for higher dimensions, though (78) is common to all dimensions.[31]

31. Geodesics.[32] The shortest lines on a surface are determined by means of the first fundamental form alone and might properly have been treated before. We shall however take them up at this point. To minimize

$$s = \int [\Sigma_{rs} a_{rs} dx_r dx_s]^{\frac{1}{2}}$$

we follow the ordinary procedure of variation:

$$\delta s = \frac{1}{ds}\int \left[\Sigma_{rs} \delta a_{rs} dx_r dx_s + 2\Sigma_{rs} a_{rs} \delta dx_r dx_s \right]$$

$$= \int \left[\frac{1}{ds} \Sigma_{rst} \frac{\partial a_{rs}}{\partial x_t} dx_r dx_s \delta x_t - 2\Sigma_{rs} d\left(a_{rs} \frac{dx_s}{ds} \right) \delta x_r \right].$$

Now by (61) and $\lambda_r = \Sigma_s a_{rs} \lambda^s$,

$$\delta s = \int ds \left[\Sigma_{rst} \frac{\partial a_{rs}}{\partial x_t} \lambda^{(r)} \lambda^{(s)} \delta x_t - 2\Sigma_r \frac{d\lambda_r}{ds} \delta x_r \right].$$

By (34)

$$\frac{d\lambda_r}{ds} = \Sigma_s \frac{d\lambda_r}{\partial x_s} \frac{dx_s}{ds} = \Sigma_s \lambda_{rs} \frac{dx_s}{ds} + \Sigma_{sqp} \begin{bmatrix} r\ s \\ q \end{bmatrix} \lambda_p a^{(pq)} \frac{dx_s}{ds}$$

$$= \Sigma_s \lambda_{rs} \lambda^{(s)} + \Sigma_{sq} \begin{bmatrix} r\ s \\ q \end{bmatrix} \lambda^{(q)} \lambda^{(s)}.$$

Hence the condition $\delta s = 0$ gives, when we set t for r in the second term and r for q in the third sum,

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31 To have a spherical representation which will generalize we should mark on the unit sphere the great circle which is the trace upon the sphere of the diametral plane parallel to the tangent plane of the surface instead of the point which is the trace of the normal. This representation would therefore be the polar of the ordinary spherical representation.

32 In this section we merely follow Ricci's *Lezioni*.

$$\Sigma_{rs}\frac{\partial a_{rs}}{\partial x_t}\lambda^{(r)}\lambda^{(s)} - 2\Sigma_s\lambda_{ts}\lambda^{(s)} - 2\Sigma_{sr}\begin{bmatrix} t\ s \\ r \end{bmatrix}\lambda^{(r)}\lambda^{(s)} = 0.$$

The first and last terms cancel and hence the condition for a geodesic in the notation of the covariant derivatives is

$$\Sigma_s\lambda_{ts}\lambda^{(s)} = 0. \tag{79}$$

In terms of the system φ_s derived from the λ's the condition is, by (64′),

$$\Sigma_s\bar{\lambda}_t\varphi_s\lambda^{(s)} = 0 \quad \text{or} \quad \Sigma_s\varphi_s\lambda^{(s)} = 0. \tag{79′}$$

The quantity $\Sigma\varphi_s\lambda^{(s)}$ is an invariant which vanishes when λ is a system of geodesics.

32. Curvature; Interpretation of $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$. The moving axis $\boldsymbol{\xi}$ is tangent to the curves λ. The curvature of these curves is $d\boldsymbol{\xi}/ds$ and from (71) takes the form

$$\frac{d\boldsymbol{\xi}}{ds} = \Sigma_r\boldsymbol{\xi}_r\frac{dx_r}{ds} = \Sigma_r\boldsymbol{\xi}_r\lambda^{(r)} = \boldsymbol{\alpha}\Sigma\lambda_r\lambda^{(r)} + \boldsymbol{\mu}\Sigma\bar{\lambda}_r\lambda^{(r)} + \boldsymbol{\eta}\Sigma\varphi_r\lambda^{(r)}.$$

Hence

$$\mathbf{c} = \frac{d\boldsymbol{\xi}}{ds} = \boldsymbol{\alpha} + \gamma\boldsymbol{\eta}, \tag{80}$$

if

$$\gamma = \Sigma\varphi_r\lambda^{(r)}, \tag{81}$$

where γ is the invariant which vanishes (as has been seen) for geodesics. *The curvature of a surface curve therefore has two components one normal to the surface and equal to the vector invariant* $\boldsymbol{\alpha}$, *one in the surface perpendicular to* λ *and of magnitude* γ. We have therefore an interpretation of the vector invariant $\boldsymbol{\alpha}$, namely, the component of the curvature perpendicular to the surface. We have also an interpretation of γ as the tangential component of the curvature. A geodesic being a curve which has no tangential component of curvature, the curvature of a geodesic is wholly normal to the surface, i. e., *the osculating plane of the geodesic is normal to the surface, no matter what the number of dimensions in which the surface lies.* We may consequently say that: *the vector* $\boldsymbol{\alpha}$ *is the curvature of the geodesic which is*

tangent to the curve λ, since $\boldsymbol{\alpha}$ depends only on the direction of the tangent $\lambda^{(r)}$ as shown by (65).

If a curve is projected on a plane (or any plane space) passing through a tangent line to the curve, the curvature of the projection at the point of tangency is equal to the projection of the curvature of the given curve at that point. To see this note first that the elements of arc on the given curve (ds) and the projected curve (ds') differ at the point of contact by infinitesimals higher than the second because their ratios involve the cosine of a small angle. The elements ds and ds' are therefore equivalent for first and second derivatives. The projection of a vector $\mathbf{r}$ on a space S_k represented by a unit vector $\mathbf{S}_k$ is

$$\mathbf{r}' = (-1)^{k-1}(\mathbf{r}\cdot\mathbf{S}_k)\cdot\mathbf{S}_k.$$

Then,

$$\mathbf{c}' = \frac{d^2\mathbf{r}'}{ds'^2} = (-1)^{k-1}\left(\frac{d^2\mathbf{r}}{ds^2}\cdot\mathbf{S}_k\right)\cdot\mathbf{S}_k = (-1)^{k-1}(\mathbf{c}\cdot\mathbf{S}_k)\cdot\mathbf{S}_k.$$

We could in like manner show that if we project a curve on a plane space through the osculating plane of the curve, the torsion of the projection is equal to projection of the torsion at that point: and so on.

We have $\mathbf{c} = \boldsymbol{\alpha} + \gamma\boldsymbol{\eta}$. If we project the curve λ on the tangent plane to the surface, we have for the curvature of the projection,

$$\begin{aligned}\mathbf{c}' &= -(\mathbf{c}\cdot\mathbf{M})\cdot\mathbf{M} = -[(\boldsymbol{\alpha}+\gamma\boldsymbol{\eta})\cdot(\boldsymbol{\xi}\times\boldsymbol{\eta})]\cdot(\boldsymbol{\xi}\times\boldsymbol{\eta})\\ &= -\gamma\boldsymbol{\xi}\cdot(\boldsymbol{\xi}\times\boldsymbol{\eta}) = \gamma\boldsymbol{\eta}.\end{aligned}$$

Hence the curvature of the projection upon the tangent plane is γ in magnitude. The invariant γ is therefore the curvature of the projection of the curve upon the tangent plane,— this is called the geodesic curvature (which must be clearly distinguished from the curvature of the geodesic tangent to the curve).

If we project on a normal plane determined by $\boldsymbol{\xi}$ and any normal $\mathbf{n}$ we have

$$\mathbf{c}' = -[(\boldsymbol{\alpha}+\gamma\boldsymbol{\eta})\cdot(\boldsymbol{\xi}\times\mathbf{n})]\cdot(\boldsymbol{\xi}\times\mathbf{n}) = (\boldsymbol{\alpha}\cdot\mathbf{n})\cdot\mathbf{n}.$$

Hence the curvature of the projection is the component of $\boldsymbol{\alpha}$ along $\mathbf{n}$. If $\mathbf{n}$ had coincided with $\boldsymbol{\alpha}$ in direction, the curvature of the projection would have been $\boldsymbol{\alpha}$.

Consider now a section of the surface by a normal space S_{n-1} of $n-1$ dimensions containing the tangent line $\boldsymbol{\xi}$ and the normal space

$\mathbf{N}_{n-2}$. The geodesic tangent to λ has for curvature $\boldsymbol{\alpha}$, as has been seen, and hence its osculating plane $\boldsymbol{\xi}\times\boldsymbol{\alpha}$ lies in S_{n-1}. The geodesic has therefore three consecutive points in S_{n-1}, i. e., to infinitesimals of the third order the geodesic coincides with the normal section and hence the curvature of the geodesic and of the normal section are equal. Consequently: *we may interpret* $\boldsymbol{\alpha}$ *as the curvature of the normal section of the surface.* So far as curvature is concerned we may replace the normal section by the geodesic.

Now $\mathbf{c} = \boldsymbol{\alpha} + \gamma\boldsymbol{\eta}$ is the curvature of any section (for the curve on the surface and the section of the surface by a space S_{n-1} containing the osculating plane of the curve are exchangeable as far as curvature is concerned) and the projection of $\mathbf{c} = \boldsymbol{\alpha} + \gamma\boldsymbol{\eta}$ on the normal is $\boldsymbol{\alpha}$ itself. Hence we have *Meusnier's theorem* that: *The projection of the curvature of any section on the normal section is the curvature of the normal section.* (Meusnier's theorem may be found in various degrees of generalization in the literature, e. g., in Levi's long article cited in note 2).

As $\mathbf{c} = \boldsymbol{\alpha} + \gamma\boldsymbol{\eta}$, $\mathbf{c}^2 = \boldsymbol{\alpha}^2 + \gamma^2$ and hence: *The magnitude of the curvature of a section is the square root of the sum of the squares of the normal and geodesic curvatures.*

33. Interpretation of $\boldsymbol{\beta}$ and $\boldsymbol{\mu}$. If we treat $d\boldsymbol{\eta}$ as we treated $d\boldsymbol{\xi}$ we find

$$\frac{d\boldsymbol{\eta}}{ds} = \boldsymbol{\mu} - \gamma\boldsymbol{\xi}. \tag{82}$$

Now $\boldsymbol{\eta}$ is a normal to the curve λ lying in the surface and $d\boldsymbol{\eta}/ds$ is the rate of change of this surface normal. If we consider the geodesic tangent to λ we have $\gamma = 0$, and hence: *The vector* $\boldsymbol{\mu}$ *(which is perpendicular to the surface) may be interpreted as the rate of change of the surface-normal to a geodesic.* In three dimensions the surface normal is the binormal of the geodesic (with the proper convention as to sign) and hence in three dimensions μ is the torsion of the geodesic tangent to λ. In higher dimensions this interpretation is no longer valid because the osculating three space of the geodesic need not contain the tangent plane **M**.

We may next form

$$\frac{d\boldsymbol{\xi}}{ds} = \Sigma_r \boldsymbol{\xi}_r \frac{\partial x_r}{d\bar{s}} = \Sigma_r \boldsymbol{\xi}_r \bar{\lambda}^{(r)} = \boldsymbol{\mu} + \boldsymbol{\eta}\Sigma_r\varphi_r\bar{\lambda}^{(r)},$$

$$\frac{d\boldsymbol{\eta}}{d\bar{s}} = \Sigma_r \boldsymbol{\eta}_r \frac{dx_r}{d\bar{s}} = \Sigma_r \boldsymbol{\eta}_r \bar{\lambda}^{(r)} = \boldsymbol{\beta} - \boldsymbol{\xi}\Sigma_r\varphi_r\lambda^{(r)},$$

by which we denote the rate of change of $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ with respect to the arc upon the orthogonal trajectories $\bar{\lambda}_r$ of λ_r.

Now the derived system φ_r for λ_r is related to the derived system $\bar{\varphi}_r$ for $\bar{\lambda}_r$ by the relation

$$\bar{\varphi}_r = -\varphi_r,$$

as may be seen from (63) and (64). Let

$$\bar{\gamma} = \Sigma_r \bar{\varphi}_r \bar{\lambda}^{(r)} = -\Sigma_r \varphi_r \bar{\lambda}^{(r)}. \tag{83}$$

Then $\bar{\gamma}$ is the geodesic curvature of the normal trajectories $\bar{\lambda}$.

$$\frac{d\boldsymbol{\xi}}{d\bar{s}} = \boldsymbol{\mu} - \bar{\gamma}\boldsymbol{\eta}, \quad \frac{d\boldsymbol{\eta}}{d\bar{s}} = \boldsymbol{\beta} + \bar{\gamma}\boldsymbol{\xi}. \tag{84}$$

By reasoning like that previously used we note that: *The vector* $\boldsymbol{\beta}$ *is the normal curvature of the orthogonal trajectories of* λ. Moreover, as the relation of $\boldsymbol{\eta}$ to $\boldsymbol{\xi}$ is the same as that of $-\boldsymbol{\xi}$ to $\boldsymbol{\eta}$ we may interpret $\boldsymbol{\mu}$ as the rate of change of the surface-normal to the geodesic tangent to λ changed in sign, that is, *the rate of change of the surface-normals tangent to normal geodesics are equal and opposite* (*vectors*). This corresponds to the theorem in three dimensions that the geodesic torsions in perpendicular directions are equal and opposite. In three dimensions, where μ is scalar the inference is immediate that there are a pair of orthogonal directions for which the geodesic torsion is zero — the lines of curvature. But in the general case $\boldsymbol{\mu}$ is a vector and may change sign without passing through zero, and we cannot affirm the existence of directions for which the rate of change of the surface-normal vanishes.

34. The mean curvature. From (68) we get,

$$\Sigma_{rs} a^{(rs)} \mathbf{y}_{rs} = \boldsymbol{\alpha} \Sigma_r \lambda_r \lambda^{(r)} + \boldsymbol{\mu} \Sigma_r (\bar{\lambda}_r \lambda^{(r)} + \lambda_r \bar{\lambda}^{(r)}) + \boldsymbol{\beta} \Sigma_r \bar{\lambda}_r \bar{\lambda}^{(r)},$$

or

$$\Sigma_{rs} a^{(rs)} \mathbf{y}_{rs} = \boldsymbol{\alpha} + \boldsymbol{\beta}. \tag{85}$$

This equation from its form on the right appears to depend on λ, but from the form on the left is seen to be independent of λ. Hence: *The vector* $\boldsymbol{\alpha} + \boldsymbol{\beta}$ *is an invariant normal vector associated with a point of the surface* — it is a special and particularly important normal selected

from all possible normals. As $\boldsymbol{\alpha}$ is the curvature of one section and $\boldsymbol{\beta}$ of the orthogonal section, we have the result that: *The sum of the normal curvatures in two orthogonal directions is independent of the directions. The sum* $\boldsymbol{\alpha} + \boldsymbol{\beta}$ *will be written as* $2\mathbf{h}$, *where* $\mathbf{h}$ *is called the mean* (*vector*) *curvature.*[33]

Since $\boldsymbol{\alpha} + \boldsymbol{\beta}$ is constant and the vector $\boldsymbol{\alpha} - \boldsymbol{\beta}$ is the other diagonal of the parallelogram on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, the vector $\boldsymbol{\alpha} - \boldsymbol{\beta}$ must pass through a fixed point on the mean curvature vector (namely, the extremity of that vector) and the termini of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ must describe a central curve about that point.

If we introduce a new pair of orthogonal directions λ' making an angle θ with λ_r we have

$$\lambda'_r = \lambda_r\cos\theta + \bar{\lambda}_r\sin\theta, \qquad \bar{\lambda}'_r = \bar{\lambda}_r\cos\theta - \lambda_r\sin\theta,$$

whence $\qquad \cos\theta = \Sigma\lambda^{(r)}\lambda'_r, \qquad \sin\theta = \Sigma\bar{\lambda}^{(r)}\lambda'_r,$

$$\lambda_r = \lambda'_r\cos\theta - \bar{\lambda}'_r\sin\theta, \qquad \bar{\lambda}_r = \lambda'_r\sin\theta + \bar{\lambda}'_r\cos\theta, \tag{86}$$
$$\lambda^{(r)} = \lambda'^{(r)}\cos\theta - \bar{\lambda}'^{(r)}\sin\theta, \qquad \bar{\lambda}^{(r)} = \lambda'^{(r)}\sin\theta + \bar{\lambda}'^{(r)}\cos\theta.$$

Now from (65) we have, in vector form,

$$\boldsymbol{\alpha} = \Sigma_{rs}\lambda^{(r)}\lambda^{(s)}\mathbf{y}_{rs}, \qquad \boldsymbol{\beta} = \Sigma_{rs}\bar{\lambda}^{(r)}\bar{\lambda}^{(s)}\mathbf{y}_{rs}, \tag{87}$$
$$\boldsymbol{\mu} = \Sigma_{rs}\bar{\lambda}^{(r)}\lambda^{(s)}\mathbf{y}_{rs} = \Sigma_{rs}\lambda^{(r)}\bar{\lambda}^{(s)}\mathbf{y}_{rs}.$$

If we substitute for the λ's in terms of the λ''s we get the relations between $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\mu}$ and $\boldsymbol{\alpha}'$, $\boldsymbol{\beta}'$, $\boldsymbol{\mu}'$ for different directions in the surface. Thus

$$\begin{aligned} \boldsymbol{\alpha} &= \boldsymbol{\alpha}'\cos^2\theta - 2\boldsymbol{\mu}'\sin\theta\cos\theta + \boldsymbol{\beta}'\sin^2\theta \\ \boldsymbol{\beta} &= \boldsymbol{\beta}'\cos^2\theta + 2\boldsymbol{\mu}'\sin\theta\cos\theta + \boldsymbol{\alpha}'\sin^2\theta \\ \boldsymbol{\mu} &= \boldsymbol{\mu}'(\cos^2\theta - \sin^2\theta) + (\boldsymbol{\alpha}' + \boldsymbol{\beta}')\sin\theta\cos\theta, \end{aligned} \tag{88}$$

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33 By mean curvature we designate the half sum of the curvatures $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. This is a true mean. In three dimensional surface theory the mean curvature often if not generally stands for the sum of the curvatures (See Eisenhart, *Differential Geometry*, page 123; E. E. Levi, loc. cit., page 69). We may quote as Levi does a theorem of Killing: the sum of the squares of the mean curvatures of the $n - 2$ three dimensional surfaces obtained by projecting an n-dimensional surface on $n - 2$ mutually perpendicular three spaces passing through the tangent plane, is constant. That is, is independent of the $n - 2$ normals selected to determine the three-spaces. The value of this invariant is $(2\mathbf{h})^2$. The theorem is of course merely the scalar form of our relation (67′).

or

$$\begin{aligned}\boldsymbol{\alpha}' &= \boldsymbol{\alpha}\cos^2\theta + 2\boldsymbol{\mu}\sin\theta\cos\theta + \boldsymbol{\beta}\sin^2\theta,\\ \boldsymbol{\beta}' &= \boldsymbol{\beta}\cos^2\theta - 2\boldsymbol{\mu}\sin\theta\cos\theta + \boldsymbol{\alpha}\sin^2\theta,\\ \boldsymbol{\mu}' &= \boldsymbol{\mu}(\cos^2\theta - \sin^2\theta) - (\boldsymbol{\alpha} - \boldsymbol{\beta})\sin\theta\cos\theta.\end{aligned} \tag{88'}$$

Hence if we write

$$\begin{aligned}\mathbf{h} &= \tfrac{1}{2}(\boldsymbol{\alpha} + \boldsymbol{\beta}) \text{ and } \boldsymbol{\delta} = \tfrac{1}{2}(\boldsymbol{\alpha} - \boldsymbol{\beta}),\\ \boldsymbol{\alpha}' &= \mathbf{h} + \boldsymbol{\mu}\sin2\theta + \boldsymbol{\delta}\cos2\theta,\\ \boldsymbol{\beta}' &= \mathbf{h} - \boldsymbol{\mu}\sin2\theta - \boldsymbol{\delta}\cos2\theta,\\ \boldsymbol{\mu}' &= \boldsymbol{\mu}\cos2\theta - \boldsymbol{\delta}\sin2\theta,\\ \boldsymbol{\delta}' &= \boldsymbol{\delta}\cos2\theta + \boldsymbol{\mu}\sin2\theta.\end{aligned} \tag{89}$$

35. The indicatrix. From equations (89) we infer that: *As θ changes, the extremity of $\boldsymbol{\alpha}'$ describes an ellipse of which $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$ are conjugate radii and of which the center is given by $\mathbf{h}$; the extremity of $\boldsymbol{\beta}$ describes the same ellipse at the opposite end of the diameter from $\boldsymbol{\alpha}$; and $\boldsymbol{\mu}'$, laid off from the center of the ellipse, describes the same ellipse, each position of $\boldsymbol{\mu}'$ being conjugate to the line joining $\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}'$ and advanced by the excentric angle $\pi/2$ from $\boldsymbol{\alpha}'$ toward $\boldsymbol{\beta}'$.* (The $\boldsymbol{\mu}$ that goes with the orthogonal trajectories is clearly $-\boldsymbol{\mu}$ as previously proved).

The conic, which we thus get, lying in the normal space, may be called the INDICATRIX. In four dimensions the whole figure including $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ lies in a plane, namely the normal plane; in higher dimensions the figure will not generally lie in a plane, the ellipse with the lines $\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}'$ forming a conical surface lying in a normal three space. *No matter how many dimensions a surface may lie in, the properties of normal curvature at any particular point may be described in a 3-space; for such properties surfaces in more than five dimensions need not be discussed.*

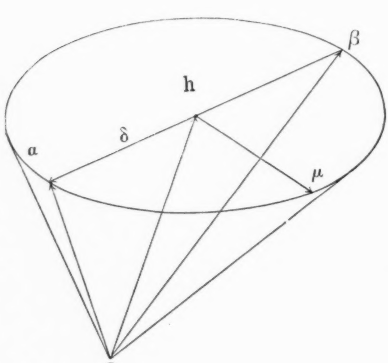

FIGURE 1.

The relation (69), that is, $\boldsymbol{\alpha}\cdot\boldsymbol{\beta} - \boldsymbol{\mu}^2 = G$, may be interpreted on our indicatrix. For

$$\boldsymbol{\alpha}\cdot\boldsymbol{\beta} = \mathbf{h}^2 - \boldsymbol{\delta}^2, \qquad \mathbf{h}^2 - (\boldsymbol{\delta}^2 + \boldsymbol{\mu}^2) = G. \tag{90}$$

Now the sum $\boldsymbol{\delta}^2 + \boldsymbol{\mu}^2$ of the squares of two conjugate radii of an ellipse is constant and equal to $a^2 + b^2$, the sum of the squares of the semi-axes. Hence: *The Gaussian invariant G is the difference of the square of the mean curvature and the sum of the squares of the semi-axes of the indicatrix.*[34]

36. Minimal surfaces.[35] The vector element of area of a surface may be written as

$$\mathbf{P}dx_1dx_2 = \frac{\partial \mathbf{y}}{\partial x_1} \times \frac{\partial \mathbf{y}}{\partial x_2}\, dx_1dx_2.$$

To find the condition for a minimal surface we write

$$0 = \delta \int\int (\mathbf{P}\cdot\mathbf{P})^{\frac{1}{2}} dx_1dx_2 = \int\int \frac{\delta\mathbf{P}\cdot\mathbf{P}}{(\mathbf{P}\cdot\mathbf{P})^{\frac{1}{2}}}\, dx_1dx_2.$$

If $\mathbf{M}$ is the unit tangent plane as heretofore, the condition becomes

$$0 = \int\int \delta\mathbf{P}\cdot\mathbf{M}dx_1dx_2,$$

$$\delta\mathbf{P} = \frac{\partial\delta\mathbf{y}}{\partial x_1} \times \frac{\partial \mathbf{y}}{\partial x_2} - \frac{\partial\delta\mathbf{y}}{\partial x_2} \times \frac{\partial \mathbf{y}}{\partial x_1}.$$

We have to integrate two terms by parts, one of which is

$$\int\int \frac{\partial\delta\mathbf{y}}{\partial x_1} \times \frac{\partial \mathbf{y}}{\partial x_2}\cdot\mathbf{M}dx_1dx_2 = -\int\int \delta\mathbf{y} \times \frac{\partial}{\partial x_1}\left(\frac{\partial \mathbf{y}}{\partial x_2}\cdot\mathbf{M}\right) dx_1dx_2,$$

omitting the integrated term which vanishes at the limits; we have then

$$\int\int \delta\mathbf{y} \times \left[\frac{\partial}{\partial x_1}\left(\frac{\partial \mathbf{y}}{\partial x_2}\cdot\mathbf{M}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial \mathbf{y}}{\partial x_1}\cdot\mathbf{M}\right)\right] dx_1dx_2 = 0.$$

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34 This result is stated by Levi, loc. cit., p. 71.

35 For special developments on minimum surfaces see Levi, loc. cit., p. 90. Eisenhart, *Amer. J. Math.*, **34**, 215–236 (1912), where references to earlier work will be found.

As $\delta \mathbf{y}$ is arbitrary we infer that the condition for a minimal surface is

$$\frac{\partial}{\partial x_1}\left(\frac{\partial \mathbf{y}}{\partial x_2}\cdot\mathbf{M}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial \mathbf{y}}{\partial x_1}\cdot\mathbf{M}\right) = 0.$$

The equation further simplifies to

$$\frac{\partial \mathbf{y}}{\partial x_2}\cdot\frac{\partial \mathbf{M}}{\partial x_1} - \frac{\partial \mathbf{y}}{\partial x_1}\cdot\frac{\partial \mathbf{M}}{\partial x_2} = 0.$$

We may use (74) and (74′) to modify the results to

$$(\boldsymbol{\xi}\lambda_2 + \boldsymbol{\eta}\bar{\lambda}_2)\cdot(\boldsymbol{\alpha}\times\boldsymbol{\eta}\lambda_1 + \boldsymbol{\mu}\times\boldsymbol{\eta}\bar{\lambda}_1 - \boldsymbol{\mu}\times\boldsymbol{\xi}\lambda_1 - \boldsymbol{\beta}\times\boldsymbol{\xi}\bar{\lambda}_1)$$
$$- (\boldsymbol{\xi}\lambda_1 + \boldsymbol{\eta}\bar{\lambda}_1)\cdot(\boldsymbol{\alpha}\times\boldsymbol{\eta}\lambda_2 + \boldsymbol{\mu}\times\boldsymbol{\eta}\bar{\lambda}_2 - \boldsymbol{\mu}\times\boldsymbol{\xi}\lambda_2 - \boldsymbol{\beta}\times\boldsymbol{\xi}\bar{\lambda}_2) = 0.$$

When we multiply the equation out we find

$$(\boldsymbol{\alpha} + \boldsymbol{\beta})\,(\bar{\lambda}_1\lambda_2 - \lambda_1\bar{\lambda}_2) = 0.$$

The term $\bar{\lambda}_1\lambda_2 - \lambda_1\bar{\lambda}_2$ cannot vanish because it is equal to $-\sqrt{a}$ as may readily be shown from the defining relations of λ and $\bar{\lambda}$;

$$\lambda_1\bar{\lambda}_2 - \bar{\lambda}_1\lambda_2 = \sqrt{a}. \tag{91}$$

Hence the condition for a minimal surface is $\boldsymbol{\alpha} + \boldsymbol{\beta} = 0$. Thus: *In any number of dimensions the condition for a minimal surface is that the mean curvature shall vanish at each point of the surface.*[36] This is the immediate generalization of the condition in three dimensions.

By reference to (78) we see that for a minimal surface, $d\mathbf{M}\cdot d\mathbf{M} = -G ds^2$. This relation in three dimensions is interpreted as showing that the spherical representation of a minimal surface is conformal: for $d\mathbf{M}\cdot d\mathbf{M} = d\mathbf{n}\cdot d\mathbf{n}$, $\mathbf{n}$ being a unit normal, and $d\mathbf{n}\cdot d\mathbf{n}$ is the differential of arc in the spherical representation. In higher dimensions we can merely say that: *The magnitude of $d\mathbf{M}/ds$ is the same for all directions through a point on a minimal surface.*

37. The intersection of consecutive normals. Let $\mathbf{N}$ be the unit normal space of $n - 2$ dimensions at any point of the surface, and $\mathbf{r}$ a vector from that point. The equation of the normal space is

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36 This result has been stated by Levi, loc. cit.

$\mathbf{r}\times\mathbf{N} = 0$. If $d\mathbf{r} = \boldsymbol{\xi}ds$ is an infinitesimal displacement along the surface and $\mathbf{N} + d\mathbf{N}$ the normal at its extremity, the equation of the adjacent normal space becomes

$$(\mathbf{r} - d\mathbf{r})\times(\mathbf{N} + d\mathbf{N}) = 0 \quad \text{or} \quad \mathbf{r}\times d\mathbf{N} - d\mathbf{r}\times\mathbf{N} = 0.$$

The intersection of the two normal spaces is determined by the simultaneous equations

$$\mathbf{r}\times\mathbf{N} = 0, \quad \mathbf{r}\times\frac{d\mathbf{N}}{ds} - \boldsymbol{\xi}\times\mathbf{N} = 0.$$

If we take complements we may write these equations

$$\mathbf{r}\cdot\mathbf{M} = 0, \quad \mathbf{r}\cdot\frac{d\mathbf{M}}{ds} - \boldsymbol{\xi}\cdot\mathbf{M} = 0. \tag{92}$$

The first equation merely states that $\mathbf{r}$ is perpendicular to $\mathbf{M}$ and we shall therefore consider only such values of $\mathbf{r}$ in the second equation. From (73), (80), (82), we have,

$$\mathbf{r}\cdot(\boldsymbol{\alpha} + \gamma\boldsymbol{\eta})\times\boldsymbol{\eta} + \boldsymbol{\xi}\times(\boldsymbol{\mu} - \gamma\boldsymbol{\xi}) - \boldsymbol{\xi}\cdot(\boldsymbol{\xi}\times\boldsymbol{\eta}) = 0,$$

or

$$\mathbf{r}\cdot(\boldsymbol{\alpha}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\mu}) + \boldsymbol{\eta} = 0,$$
$$-\mathbf{r}\cdot\boldsymbol{\alpha}\boldsymbol{\eta} + \mathbf{r}\cdot\boldsymbol{\mu}\boldsymbol{\xi} = -\boldsymbol{\eta}.$$

Hence

$$\mathbf{r}\cdot\boldsymbol{\alpha} = 1, \quad \mathbf{r}\cdot\boldsymbol{\mu} = 0. \tag{92'}$$

Special Cases. Consider first the case $n = 4$. Here the indicatrix is a conic in a plane through the surface-point O. The vector $\boldsymbol{\alpha}$ runs from O to a point of this conic. If we lay off from O the radius of curvature instead of the curvature itself, we get a point Q which is the inverse of P with respect to O. The locus of Q is therefore a bicircular quartic. If we draw through Q a line perpendicular to $\boldsymbol{\alpha}$, we have a line for which $\mathbf{r}\cdot\boldsymbol{\alpha} = 1$; and the point where this line cuts the perpendicular from O upon $\boldsymbol{\mu}$, or upon the tangent to the indicatrix at P, is a point P' which is the common solution of (92′) and which therefore is the point of intersection of the normal plane $\mathbf{N}$ with the adjacent normal plane in the direction $\boldsymbol{\xi}$.

If we consider the triangles OPM and OQP' we see that $OM\cdot OP' = OQ\cdot OP = 1$. Hence P' and M are inverse points. But the locus

of M is the pedal of the indicatrix and hence we have the theorem: *The inverse of the pedal of the indicatrix is the locus of points where consecutive normal planes about a point intersect the normal plane at the point.*[37]

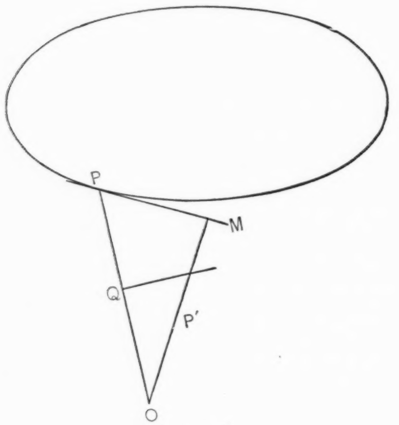

FIGURE 2.

Consider next the case $n = 5$. Here the indicatrix is a conic which may or may not lie in a plane through O. In the latter special case the reasoning before holds except for the fact that the solution for $\mathbf{r}$ in (92′) is no longer a point, but a line through that point perpendicular to the plane of the conic. The locus of intersection of consecutive normal spaces is therefore a right cylinder of which the directrix is the conic which is the inverse of the pedal of the indicatrix. This is merely a direct extension of the case previously treated.

The general case. If the indicatrix does not lie in a plane with O, and if we lay off along $\boldsymbol{\alpha}$ the distance equal to the radius of curvature, instead of equal to the curvature, we get a point Q which lies both on the cone determined by O as vertex and the indicatrix as directrix and on the sphere through O which is the inverse of the plane of the indicatrix. The locus of Q is therefore a sphero-conic. The plane $\mathbf{r}\cdot\boldsymbol{\alpha} = 1$ passes through the point Q and is perpendicular to $\boldsymbol{\alpha}$; it therefore passes through the point O' of the sphere diametrically opposite to O, this point O' being also the inverse of the foot F of the perpendicular OF from O upon the plane of the indicatrix.

Now $\mathbf{r}\cdot\boldsymbol{\mu} = 0$ is the plane through O perpendicular to $\boldsymbol{\mu}$, and hence perpendicular to the plane of the indicatrix, and hence finally $\mathbf{r}\cdot\boldsymbol{\mu} = 0$ is a plane through the line OF. The intersection of $\mathbf{r}\cdot\boldsymbol{\mu} = 0$ and $\mathbf{r}\cdot\boldsymbol{\alpha} = 1$ is therefore a line through O' perpendicular alike to $\boldsymbol{\mu}$ and $\boldsymbol{\alpha}$, and consequently perpendicular to the plane tangent to the cone (described by $\boldsymbol{\alpha}$) through the element $\boldsymbol{\alpha}$ (since $\boldsymbol{\mu}$ is parallel to the tangent to the indicatrix at the extremity of $\boldsymbol{\alpha}$).

37 Kommerell, loc. cit.

Cones I and II. We may now state that: *The locus of intersection of consecutive normal spaces N_3 generate a cone (thus all the consecutive normal spaces pass through a point). This cone is a quadric cone.* For if $\mathbf{r}\cdot\Phi^{-1}\cdot\mathbf{r} = 0$, where Φ^{-1} is a self conjugate dyadic, be the equation of the cone described by $\boldsymbol{\alpha}$, which we shall call Cone I, the normal to the tangent plane is determinable from the equation $\mathbf{r}\cdot\Phi^{-1}\cdot d\mathbf{r} = 0$ and $\mathbf{r}\cdot\Phi^{-1} = \mathbf{n}$ or $\mathbf{r} = \Phi\cdot\mathbf{n}$. Hence the locus of $\mathbf{n}$ is $\mathbf{n}\cdot\Phi\cdot\mathbf{n} = 0$, the reciprocal cone to Cone I. This reciprocal cone we shall call Cone II; its vertex is at O', not at O. *As the Cones* I *and* II *are reciprocal, we can infer not only that the normals to the tangent planes to* I *generate* II *but that reciprocally the normals to the tangent planes to* II *generate* I. The special case previously treated where O lies in the plane of the indicatrix falls under the general case because O' has retreated to infinity and consequently Cone II becomes a cylinder.

In case $n > 5$ we may, if we desire, restrict ourselves to the normal space N_3 in which the indicatrix lies. We shall then have precisely the relations just proved for the case $n = 5$. But when $n > 5$ the equations (92′) have additional solutions in the rest of the total normal space N_{n-2} external to the particular N_3. *The adjacent normal spaces N_{n-2} intersect in a space N_{n-4} which is perpendicular to N_3 and contains in N_3 an element in Cone* II.

38. The fundamental dyadic Φ. The forms of Φ and Φ^{-1} which determine Cones II and I may be found in the general case as follows. Let $\mathbf{h}'$, $\boldsymbol{\mu}'$, $\boldsymbol{\delta}'$ be the reciprocal set to $\mathbf{h}$, $\boldsymbol{\mu}$, and $\boldsymbol{\delta}$, and consider

$$\Phi = c(\mathbf{hh} - \boldsymbol{\mu\mu} - \boldsymbol{\delta\delta}), \quad \Phi^{-1} = c^{-1}(\mathbf{h}'\mathbf{h}' - \boldsymbol{\mu}'\boldsymbol{\mu}' - \boldsymbol{\delta}'\boldsymbol{\delta}'),$$

where c is any constant. The vectors $\mathbf{h} + \boldsymbol{\delta}$ lie on $\mathbf{r}\cdot\Phi^{-1}\cdot\mathbf{r} = 0$. But

$$(\mathbf{h} + \boldsymbol{\delta})\cdot(\mathbf{h}'\mathbf{h}' - \boldsymbol{\mu}'\boldsymbol{\mu}' - \boldsymbol{\delta}'\boldsymbol{\delta}')\cdot(\mathbf{h} + \boldsymbol{\delta}) = 0.$$

Moreover the expression $\boldsymbol{\mu\mu} + \boldsymbol{\delta\delta}$ is invariant of the system λ as may be seen from (89) where the accents denote new values of $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$ not the reciprocal set as here. As $\mathbf{h}$ is independent of λ, the diadics Φ and Φ^{-1} are independent of λ and any value of $\mathbf{h} + \boldsymbol{\delta}$ will satisfy $\mathbf{r}\cdot\Phi^{-1}\cdot\mathbf{r} = 0$. The expression written down for Φ is therefore correct.

The dyadic Φ may be expressed directly and simply in terms of the vector coefficients of the form Ψ. Consider the dyadic Ω which is the discriminant matrix of Ψ, namely,

$$\Omega = \begin{vmatrix} \mathbf{y}_{11} & \mathbf{y}_{12} \\ \mathbf{y}_{21} & \mathbf{y}_{22} \end{vmatrix} = \mathbf{y}_{11}\mathbf{y}_{22} - \mathbf{y}_{12}\mathbf{y}_{21}. \tag{93}$$

As $\mathbf{y}_{11}\mathbf{y}_{22} \neq \mathbf{y}_{22}\mathbf{y}_{11}$ this dyadic is not self conjugate.

$$\begin{aligned}\Omega &= [\boldsymbol{\alpha}\lambda_1^2 + 2\boldsymbol{\mu}\lambda_1\bar{\lambda}_1 + \boldsymbol{\beta}\bar{\lambda}_1^2][\boldsymbol{\alpha}\lambda_2^2 + 2\boldsymbol{\mu}\lambda_2\bar{\lambda}_2 + \boldsymbol{\beta}\bar{\lambda}_2^2] \\ &\quad - [\boldsymbol{\alpha}\lambda_1\lambda_2 + \boldsymbol{\mu}(\lambda_1\bar{\lambda}_2 + \lambda_2\bar{\lambda}_1) + \boldsymbol{\beta}\bar{\lambda}_1\bar{\lambda}_2]^2 \\ &= [-\boldsymbol{\mu}\boldsymbol{\mu}(\lambda_1\bar{\lambda}_2 - \lambda_2\bar{\lambda}_1) + \boldsymbol{\alpha}\boldsymbol{\beta}\lambda_1\bar{\lambda}_2 - \boldsymbol{\beta}\boldsymbol{\alpha}\bar{\lambda}_1\lambda_2 \\ &\quad + (\boldsymbol{\alpha}\boldsymbol{\mu} - \boldsymbol{\mu}\boldsymbol{\alpha})\lambda_1\lambda_2 + (\boldsymbol{\mu}\boldsymbol{\beta} - \boldsymbol{\beta}\boldsymbol{\mu})\bar{\lambda}_1\bar{\lambda}_2](\lambda_1\bar{\lambda}_2 - \lambda_2\bar{\lambda}_1).\end{aligned}$$

The first term is self-conjugate and the last two are anti-self-conjugate.

$$\begin{aligned}\tfrac{1}{2}(\Omega + \Omega_C) &= [\tfrac{1}{2}(\boldsymbol{\alpha}\boldsymbol{\beta} + \boldsymbol{\beta}\boldsymbol{\alpha}) - \boldsymbol{\mu}\boldsymbol{\mu}](\lambda_1\bar{\lambda}_2 - \lambda_2\bar{\lambda}_1)^2 \\ &= [\mathbf{hh} - \boldsymbol{\mu}\boldsymbol{\mu} - \boldsymbol{\delta}\boldsymbol{\delta}]a,\end{aligned}$$

by (91). We find therefore that: *The selfconjugate part of the vector matrix* Ω *is the dyadic* Φ *which defines Cone* II, with the multiplier $c = a$. We shall use for Φ the value

$$\Phi = [\mathbf{hh} - \boldsymbol{\mu}\boldsymbol{\mu} - \boldsymbol{\delta}\boldsymbol{\delta}]a, \quad \text{Cone II}, \tag{94}$$

including the multiplier a; and for Φ^{-1},

$$\Phi^{-1} = [\mathbf{h}'\mathbf{h}' - \boldsymbol{\mu}'\boldsymbol{\mu}' - \boldsymbol{\delta}'\boldsymbol{\delta}']a^{-1}, \quad \text{Cone I}. \tag{94'}$$

The value of the scalar invariant Ω_S of the dyadic Ω and of the self-conjugate part of Ω are the same. Hence,

$$\Omega_S = \Phi_S = \mathbf{y}_{11}\cdot\mathbf{y}_{22} - \mathbf{y}_{12}^2 = (\mathbf{h}^2 - \boldsymbol{\mu}^2 - \boldsymbol{\delta}^2)a = Ga. \tag{95}$$

We have therefore the result that: *The Gaussian curvature G is*

$$G = \frac{\Omega_S}{a} = \frac{\mathbf{y}_{11}\cdot\mathbf{y}_{22} - \mathbf{y}_{12}^2}{a}, \tag{95'}$$

the quotient of the scalar of the matrix of the second fundamental form by the discriminant of the first fundamental form, in complete analogy with the result in three dimensions which expresses G as the quotient of the determinants of the two fundamental forms. It has already been seen that the mean curvature $\mathbf{h}$ is expressed as

$$2\mathbf{h} = \Sigma_{ij}a^{(ij)}\mathbf{y}_{ij}$$

in conformity with the expression for the analogous quantity in three dimensions.

The plane $\boldsymbol{\delta}\times\boldsymbol{\mu}$ of the indicatrix is polar to $\mathbf{h}$ with respect to Cone I, that is, it is perpendicular to $\Phi^{-1}\cdot\mathbf{h}$, as may also be seen by direct substitution in (94′).

The second fundamental forms ψ_i of the projection of the surface on any 3-space containing the tangent plane $\mathbf{M}$ at a point and some normal $\mathbf{z}_i$ is $\psi_i = \mathbf{z}_i\cdot\Psi$. If we write,

$$\Psi = \Sigma\mathbf{z}_i\psi_i = \Sigma\mathbf{z}_i\mathbf{z}_i\cdot\Psi = \mathrm{I}\cdot\Psi,$$

the mean curvature of the surface is seen to be the vector sum of the mean curvatures of the projections on the spaces determined successively by $\mathbf{z}_i$, namely,

$$2\mathbf{h} = \Sigma_{ij}a^{(ij)}\mathbf{y}_{ij} = \Sigma_{ijk}(a^{(ij)}\mathbf{y}_{ij}\cdot\mathbf{z}_k)\mathbf{z}_k.$$

There is no need of letting k vary over more than the values 1, 2, 3, as curvature phenomena are five dimensional. The expression for G may be written

$$\begin{aligned} aG = \Omega_S = \Omega{:}\mathrm{I} = \Phi{:}\mathrm{I} &= \mathbf{y}_{11}\cdot\mathrm{I}\cdot\mathbf{y}_{22} - \mathbf{y}_{12}\cdot\mathrm{I}\cdot\mathbf{y}_{21} \\ &= \Sigma_k(\mathbf{y}_{11}\cdot\mathbf{z}_k\mathbf{z}_k\cdot\mathbf{y}_{22} - \mathbf{y}_{12}\cdot\mathbf{z}_k\mathbf{z}_k\cdot\mathbf{y}_{21}). \end{aligned}$$

As the individual parentheses here are the values of G for the projections of the surface it shows that: *The total curvature of a surface is the algebraic sum of the total curvatures of the orthogonal projections of the surface.*

Since $aG = \Phi{:}(\mathbf{z}_1\mathbf{z}_1 + \mathbf{z}_2\mathbf{z}_2 + \mathbf{z}_3\mathbf{z}_3)$ we may reduce aG to a single term by choosing $\mathbf{z}_2$ and $\mathbf{z}_3$ on the cone $\mathbf{r}\cdot\Phi\cdot\mathbf{r} = 0$, i. e., upon Cone II. Then $aG = \mathbf{z}_1\cdot\Phi\cdot\mathbf{z}_1$. As $aG = \Phi_S$, this relation may be written as,

$$\Phi_S\mathbf{z}_1\cdot\mathbf{z}_1 = \mathbf{z}_1\cdot\Phi\cdot\mathbf{z}_1$$

or

$$\mathbf{z}_1\cdot(\Phi_S\mathrm{I} - \Phi)\cdot\mathbf{z}_1 = 0.$$

We therefore have another cone,

$$\mathbf{r}\cdot(\Phi_S\mathrm{I} - \Phi)\cdot\mathbf{r} = 0, \qquad \text{Cone III}, \tag{96}$$

which is coaxial with Cones I and II and which has the property that if one normal $\mathbf{z}_i$ lies upon it, the other two may lie upon Cone II, and

G will reduce to a single term. In other words: *It is possible in* ∞^1 *ways so to select three perpendicular normals* $\mathbf{z}_i$ *that one of the projections has the entire total curvature of the surface and the other two have zero total curvatures.*

If now the dyadics Φ, Φ^{-1}, $\Phi_S \mathrm{I} - \Phi = \Xi$, be referred to their principal directions,

$$\begin{aligned} \Phi^{-1} &= \frac{\mathbf{ii}}{a^2} + \frac{\mathbf{jj}}{b^2} - \frac{\mathbf{kk}}{c^2}, \\ \Phi &= a^2\mathbf{ii} + b^2\mathbf{jj} - c^2\mathbf{kk}, \\ \Xi &= (b^2 - c^2)\mathbf{ii} + (a^2 - c^2)\mathbf{jj} + (a^2 + b^2)\mathbf{kk}, \end{aligned} \tag{97}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, stand for $\mathbf{z}_1$, $\mathbf{z}_2$, $\mathbf{z}_3$. If c^2 is less than a^2 and b^2 the cone defined by Ξ is not real and the above resolution of the surface is impossible in the real domain,— as must be expected when Cone II is so narrow as to have no vertical angles as great as 90°. In case $G = 0$, $\Phi_S = 0$ and Cones II and III coincide. Hence: *The condition* $G = 0$ *implies that Cone II is a cone circumscribed about a trirectangular trihedral angle.*

As Cone I is reciprocal to Cone II, Cone I in that case must be inscriptable in a trirectangular trihedral angle or: *when* $G = 0$, *the indicatrix must be tangent to three mutually perpendicular planes through the surface-point.* In the special case where the indicatrix lies in a plane through the surface point the condition requires that the conic subtend an angle of 90° at the surface point or that the surface point must lie upon a circle of radius $(a^2 + b^2)^{\frac{1}{2}}$ concentric with the indicatrix.

39. The scalar invariants. We have now interpretations for two fundamental invariants, G and h, and the expressions of these invariants in terms of the coefficients a_{ij} and $\mathbf{y}_{ij}$. The indicatrix and its position relative to the origin require for their determination, apart from the rotation in space, five invariant scalars as remarked by Levi (see note 27). The dyadic Φ has of course three invariants Φ_S, Φ_{2S}, Φ_3 which are the coefficients in the characteristic equation for Φ. Of these the last is $\Phi_3 = a^3(\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h})^2$ as in (101). *The geometrical meaning of* Φ_3 *is, except for a factor, the square of the volume of the cone intercepted by the plane of the indicatrix from the infinite surface of Cone* I. Except for a factor this is Levi's invariant Δ_5; Φ_S is his Δ_1; and h his Δ_2.

The values of Φ_2 and $\boldsymbol{\mu}\times\boldsymbol{\delta}$ may be found in terms of $\mathbf{y}_{ij}$ as follows. From (93) and (94),

$$\Phi = (\mathbf{hh} - \boldsymbol{\mu\mu} - \boldsymbol{\delta\delta})a = \tfrac{1}{2}\mathbf{y}_{11}\mathbf{y}_{22} + \tfrac{1}{2}\mathbf{y}_{22}\mathbf{y}_{11} - \mathbf{y}_{12}\mathbf{y}_{21}\,, \tag{99}$$

$$\begin{aligned}\Phi_2 &= (\boldsymbol{\mu}\times\boldsymbol{\delta}\;\boldsymbol{\mu}\times\boldsymbol{\delta} - \boldsymbol{\delta}\times\mathbf{h}\;\boldsymbol{\delta}\times\mathbf{h} - \mathbf{h}\times\boldsymbol{\mu}\;\mathbf{h}\times\boldsymbol{\mu})a^2 \\ &= (-\tfrac{1}{4}\mathbf{y}_{22}\times\mathbf{y}_{11}\mathbf{y}_{22}\times\mathbf{y}_{11} + \tfrac{1}{2}\mathbf{y}_{11}\times\mathbf{y}_{12}\mathbf{y}_{12}\times\mathbf{y}_{22} + \tfrac{1}{2}\mathbf{y}_{12}\times\mathbf{y}_{22}\mathbf{y}_{11}\times\mathbf{y}_{12}),\end{aligned} \tag{100}$$

$$\Phi_3 = (\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h})^2a^3 = \tfrac{1}{4}(\mathbf{y}_{11}\times\mathbf{y}_{12}\times\mathbf{y}_{22})^2, \tag{101}$$

where Φ_2 and Φ_3 represent the Gibbs's double products.[38] Now

$$\Phi_2\times\mathbf{h} = (\boldsymbol{\mu}\times\boldsymbol{\delta}\;\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h})a^2, \qquad 2\mathbf{h} = \Sigma a^{(rs)}\mathbf{y}_{rs}\,,$$

$$\Phi_2\times\mathbf{h} = (-\tfrac{1}{4}a^{(12)}\mathbf{y}_{22}\times\mathbf{y}_{11} + \tfrac{1}{4}a^{(11)}\mathbf{y}_{11}\times\mathbf{y}_{12} + \tfrac{1}{4}a^{(22)}\mathbf{y}_{12}\times\mathbf{y}_{22})(\mathbf{y}_{11}\times\mathbf{y}_{12}\times\mathbf{y}_{22}).$$

Choose,

$$\pm\, a^{\frac{1}{2}}\boldsymbol{\mu}\times\boldsymbol{\delta} = -\tfrac{1}{2}a^{(12)}\mathbf{y}_{22}\times\mathbf{y}_{11} + \tfrac{1}{2}a^{(11)}\mathbf{y}_{11}\times\mathbf{y}_{12} + \tfrac{1}{2}a^{(22)}\mathbf{y}_{12}\times\mathbf{y}_{22}\,.$$

Then, $$\begin{aligned} a\boldsymbol{\mu}\times\boldsymbol{\delta}\;\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h} &= (-\tfrac{1}{2}a^{(12)}\mathbf{y}_{22}\times\mathbf{y}_{11} + \tfrac{1}{2}a^{(11)}\mathbf{y}_{11}\times\mathbf{y}_{12} + \tfrac{1}{2}a^{(22)}\mathbf{y}_{12}\times\mathbf{y}_{22}) \\ &\quad (-\tfrac{1}{2}a^{(12)2} + \tfrac{1}{4}a^{(22)}a^{(11)} + \tfrac{1}{4}a^{(11)}a^{(22)})\mathbf{y}_{11}\times\mathbf{y}_{12}\times\mathbf{y}_{22},\end{aligned}$$

and the result checks.

The double sign which arises here has come in through the extraction of a root. We may obtain from (87) the value of $\boldsymbol{\mu}\times\boldsymbol{\delta}$ as follows;

$$\boldsymbol{\mu}\times\boldsymbol{\delta} = \frac{1}{2}\Sigma_{rs}\bar{\lambda}^{(r)}\lambda^{(s)}\mathbf{y}_{rs}\times[\Sigma_{pq}(\lambda^{(p)}\lambda^{(q)} - \bar{\lambda}^{(p)}\bar{\lambda}^{(q)})\mathbf{y}_{pq}]\,.$$

The coefficient of $\mathbf{y}_{22}\times\mathbf{y}_{11}$ is

$$\bar{\lambda}^{(2)}\lambda^{(2)}(\lambda^{(1)2} - \bar{\lambda}^{(1)2}) - \bar{\lambda}^{(1)}\lambda^{(1)}(\lambda^{(2)2} - \bar{\lambda}^{(2)2}),$$

which by virtue of (61′) and (63) reduces to $a_{12}/a^{\frac{3}{2}}$. The sign of the term is therefore plus. In like manner the sign of $\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h}$ may be determined. Hence

$$\begin{aligned} 2a^{\frac{3}{2}}\boldsymbol{\mu}\times\boldsymbol{\delta} &= (a_{12}\mathbf{y}_{22}\times\mathbf{y}_{11} + a_{22}\mathbf{y}_{11}\times\mathbf{y}_{12} + a_{11}\mathbf{y}_{12}\times\mathbf{y}_{22}), \\ 2a^{\frac{3}{2}}\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h} &= \mathbf{y}_{11}\times\mathbf{y}_{12}\times\mathbf{y}_{22}\,.\end{aligned} \tag{102}$$

38 See Gibbs-Wilson, *Vector Analysis*, p. 306. As we are using the progressive product $\Phi_3 = \frac{1}{6}\,\Phi{}^{\times}_{\times}\Phi{}^{\times}_{\times}\Phi$ instead of $\frac{1}{6}\,\Phi{}^{\times}_{\times}\Phi : \Phi$. See also Wilson, *Trans. Conn. Acad., New Haven*, **14**, 1–57 (1908).

The conditions $\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h} = 0$ and $\mathbf{y}_{11}\times\mathbf{y}_{12}\times\mathbf{y}_{22}$ are therefore equivalent as was to be expected. If we use an orthogonal system of curves for the parameter curves, $a_{12} = 0$, and $\boldsymbol{\mu}\times\boldsymbol{\delta}$ may be factored. If we use a minimum system, $a_{11} = a_{22} = 0$, and $\boldsymbol{\mu}\times\boldsymbol{\delta}$ reduces to $\mathbf{y}_{11}\times\mathbf{y}_{22}$ except for a factor. In general $\boldsymbol{\mu}\times\boldsymbol{\delta}$ may be factored in ∞^3 ways of which one simple case is,

$$2a^{\frac{3}{2}}\boldsymbol{\mu}\times\boldsymbol{\delta} = \left(a_{12}\frac{\mathbf{y}_{11} + \mathbf{y}_{22}}{a_{11} + a_{22}} - \mathbf{y}_{12}\right) \times (a_{22}\mathbf{y}_{11} - a_{11}\mathbf{y}_{22}).$$

The vertex of Cone II is located at the point,

$$\boldsymbol{\nu} = \frac{(\boldsymbol{\mu}\times\boldsymbol{\delta})\cdot(\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h})}{(\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h})^2}, \tag{103}$$

which may be expressed in terms of the $\mathbf{y}$'s if desired.

The invariant $[\boldsymbol{\mu}\times\boldsymbol{\delta}]^2$ which is proportional to the square of the area of the indicatrix is except for a factor Levi's invariant Δ_4. The invariant

$$\Phi_{2S} = \mathbf{y}_{11}\times\mathbf{y}_{12}\cdot\mathbf{y}_{12}\times\mathbf{y}_{22} - \frac{1}{4}[\mathbf{y}_{22}\times\mathbf{y}_{11}]^2 \tag{104}$$

is, except for a factor, Levi's invariant Δ_3. We have geometric interpretations for all the invariants except Φ_{2S}. If we write

$$\Phi_{2S}/a^2 = [\boldsymbol{\mu}\times\boldsymbol{\delta}]^2 - [\boldsymbol{\delta}\times\mathbf{h}]^2 - [\mathbf{h}\times\boldsymbol{\mu}]^2, \tag{104'}$$

we have $\frac{1}{2}[\boldsymbol{\mu}\times\boldsymbol{\delta}]$ interpretable as the area of the triangle of which the conjugate radii $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$ are sides; $\frac{1}{2}[\boldsymbol{\delta}\times\mathbf{h}]$ as the area of the triangle of which $\boldsymbol{\delta}$ and $\mathbf{h}$ or $\boldsymbol{\delta}$ and $\boldsymbol{\alpha}$ are sides; $\frac{1}{2}[\mathbf{h}\times\boldsymbol{\mu}]$ as the area of the triangle of which $\mathbf{h}$ and $\boldsymbol{\mu}$ are sides. As $[\boldsymbol{\mu}\times\boldsymbol{\delta}]^2$ is itself an invariant $[\boldsymbol{\mu}\times\boldsymbol{\delta}]^2 - \Phi_{2S}/a^2$ is an invariant and is equal to four times the sum of the squares of the areas of the triangles on $\boldsymbol{\delta}$ and $\mathbf{h}$ and on $\mathbf{h}$ and $\boldsymbol{\mu}$.

We can therefore set up the following list of five scalar invariants,[39]

$$\mathbf{h}^2,\quad \Phi_S/a = G,\quad [\boldsymbol{\mu}\times\boldsymbol{\delta}]^2,\quad [\boldsymbol{\mu}\times\boldsymbol{\delta}]^2 - \Phi_{2S}/a^2,\quad [\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h}]^2.$$

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39 To aid the reader to make the comparison between our notation and Levi's we give the following table of equivalents for his symbols I and J.

$I_{1010} = a_{11},\quad I_{1001} = a_{12},\quad I_{0101} = a_{22},$
$J_{2020} = \mathbf{y}_{11}^2,\quad J_{1111} = \mathbf{y}_{12}^2,\quad J_{0202} = \mathbf{y}_{22}^2,$
$J_{2002} = \mathbf{y}_{11}\cdot\mathbf{y}_{22},\quad J_{2021} = \mathbf{y}_{11}\cdot\mathbf{y}_{12},\quad J_{0211} = \mathbf{y}_{22}\cdot\mathbf{y}_{12}.$

The condition on the surface due to the vanishing of these invariants is as follows:

(1) $\mathbf{h}^2 = 0$, minimum surfaces (§36),

(2) $G = 0$, developable surfaces (§41),

(3) $[\boldsymbol{\mu}\times\boldsymbol{\delta}]^2 - \Phi_{2S}/a^2 = 0$, surfaces with what Levi calls axial points, viz., three dimensional surfaces or surfaces formed by the tangents to a twisted curve (§43).

(4) $[\boldsymbol{\mu}\times\boldsymbol{\delta}]^2 = 0$, surfaces with perpendicular (Segre) characteristics — the indicatrix reduces to a linear segment — the simplest generalization of ordinary surfaces (§43).

(5) $[\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h}]^2 = 0$, surfaces possessing (Segre) characteristics — surfaces with what Levi calls planar points (§43).

Conditions (1), (3), (4) imply (5); condition (3) implies (4). We have already discussed minimum surfaces briefly; we shall take up the other types in some detail in later sections.

CHAPTER III. SPECIAL DEVELOPMENTS IN SURFACE THEORY.

40. The twisted curve surfaces and ruled surfaces. As an illustration and application of the foregoing analysis, we may treat the case of the surface formed by the tangents to a twisted curve in n dimensions. Let $\mathbf{y} = \mathbf{f}(u)$ be the equation of the curve, u being the arc. The surface is then,

$$\begin{aligned} \mathbf{y} &= \mathbf{f}(u) + v\mathbf{f}'(u), \quad \mathbf{f}'\cdot\mathbf{f}' = 1, \quad \mathbf{f}'\cdot\mathbf{f}'' = 0, \\ d\mathbf{y} &= (\mathbf{f}' + v\mathbf{f}'')du + \mathbf{f}'dv, \\ ds^2 &= d\mathbf{y}\cdot d\mathbf{y} = (\mathbf{f}' + v\mathbf{f}'')^2du^2 + 2du\,dv + dv^2 \\ &= (1 + v^2/R^2)du^2 + 2du\,dv + dv^2, \end{aligned}$$

where $R = (\mathbf{f}''\cdot\mathbf{f}'')^{-\frac{1}{2}}$ is the radius of curvature of the curve. That *the surface is developable* follows from the familiar argument, namely: ds does not depend upon the torsion of the curve and hence the surface is applicable upon the tangent surface to all curves for which R is the same function of u, and a plane curve can be found satisfying this condition.

To calculate Ψ two methods are available based on (75) and (76). The advantage of the first form (75) is that the expression $d\mathbf{y}\times d\mathbf{M}$ may be replaced by $(d\mathbf{y}\times dU\mathbf{M})/U$, where U is any scalar function;—since $d\mathbf{y}$ lies in $\mathbf{M}$ and $d\mathbf{y}\times\mathbf{M} = 0$. Now

$$\begin{aligned} \mathbf{M} &= \frac{(\mathbf{f}' + v\mathbf{f}'')\times\mathbf{f}'}{\sqrt{[(\mathbf{f}' + v\mathbf{f}'')\times\mathbf{f}']^2}} = \frac{\mathbf{f}''\times\mathbf{f}'}{\sqrt{\mathbf{f}''\cdot\mathbf{f}''}} = R\mathbf{f}''\times\mathbf{f}', \\ d\mathbf{y}\times d\mathbf{M} &= R(\mathbf{f}' + v\mathbf{f}'')\times\mathbf{f}'''\times\mathbf{f}'du, \\ &= Rv\mathbf{f}''\times\mathbf{f}'''\times\mathbf{f}'du = -\mathbf{M}\times\Psi. \end{aligned}$$

Multiply by $\mathbf{M}$ as in the text and repeat the argument there given. Then

$$\begin{aligned} -\Psi &= R^2v(\mathbf{f}''\times\mathbf{f}')\cdot(\mathbf{f}''\times\mathbf{f}'''\times\mathbf{f}')du^2 \\ &= R^2v[\mathbf{f}''\cdot\mathbf{f}'''\mathbf{f}'' - \mathbf{f}''\cdot\mathbf{f}''\mathbf{f}''' + \mathbf{f}''\cdot\mathbf{f}''\mathbf{f}'\cdot\mathbf{f}'''\mathbf{f}']du^2, \\ \Psi &= v[\mathbf{f}''' + \mathbf{f}'/R^2 - \mathbf{f}''\cdot\mathbf{f}'''\mathbf{f}'']du^2, \end{aligned}$$

since $$\mathbf{f}'\cdot\mathbf{f}''' + \mathbf{f}''\cdot\mathbf{f}'' = 0.$$

Hence comparing with $\Psi = \Sigma \mathbf{y}_{rs} dx_r dx_s$,

$$\mathbf{y}_{11} = v[\mathbf{f}''' + \mathbf{f}'/R^2 - \mathbf{f}''\cdot\mathbf{f}'''\mathbf{f}'']$$

$$\mathbf{y}_{12} = 0, \qquad \mathbf{y}_{22} = 0.$$

Also, comparing ds^2 with its standard form,

$$a_{11} = 1 + v^2/R^2, \quad a_{21} = 1, a_{22} = 1, \quad a = v^2/R^2,$$

$$a^{(11)} = R^2/v^2, \quad a^{(12)} = - R^2/v^2, \quad a^{(22)} = 1 + R^2/v^2,$$

$$2\mathbf{h} = \Sigma a^{(rs)}\mathbf{y}_{rs} = R^2 v^{-1}[\mathbf{f}''' + \mathbf{f}'/R^2 - \mathbf{f}''\cdot\mathbf{f}''\mathbf{f}''],$$

$$\Phi = \tfrac{1}{2}(\mathbf{y}_{11}\mathbf{y}_{22} + \mathbf{y}_{22}\mathbf{y}_{11}) - \mathbf{y}_{12}\mathbf{y}_{21} = 0.$$

The dyadic Φ vanished identically. Hence $\mathbf{hh} = \boldsymbol{\mu\mu} + \boldsymbol{\delta\delta}$, and $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$ must be collinear with $\mathbf{h}$. *The indicatrix for a twisted curve surface reduces to a line along the vector* $\mathbf{h}$, *extending from the surface* (vertex of degenerate Cone I) *to the end of* $2\mathbf{h}$. As $\Phi_S = 0$, the condition $G = 0$, is satisfied, as must be the case from the reasoning given at the outset.

By a similar method we may calculate the various quantities arising in the case of any surface expressed in parametric form as $\mathbf{y} = \mathbf{y}(u, v)$. Let

$$d\mathbf{y} = \mathbf{m}du + \mathbf{n}dv, \quad \mathbf{m} = \partial\mathbf{y}/\partial u, \quad \mathbf{n} = \partial\mathbf{y}/\partial v;$$

$$ds^2 = d\mathbf{y}\cdot d\mathbf{y} = \mathbf{m}^2du^2 + 2\mathbf{m}\cdot\mathbf{n}dudv + \mathbf{n}^2dv^2;$$

$$a_{11} = \mathbf{m}^2, \quad a_{12} = \mathbf{m}\cdot\mathbf{n}, \quad a_{22} = \mathbf{n}^2,$$

$$a = a_{11}a_{22} - a_{12}{}^2 = \mathbf{m}^2\mathbf{n}^2 - (\mathbf{m}\cdot\mathbf{n})^2 = (\mathbf{m}\times\mathbf{n})^2;$$

$$\mathbf{M} = \frac{\mathbf{m}\times\mathbf{n}}{\sqrt{a}} d\mathbf{y}\times d\mathbf{M} = - \mathbf{m}\times\mathbf{n}\times\frac{d\mathbf{m}du + d\mathbf{n}dv}{\sqrt{a}}.$$

Let $$d\mathbf{m} = \mathbf{p}du + \mathbf{q}dv, \quad d\mathbf{n} = \mathbf{q}du + \mathbf{r}dv,$$

$$\mathbf{p} = \partial^2\mathbf{y}/\partial u^2, \quad \mathbf{q} = \partial^2\mathbf{y}/\partial u\partial v, \quad \mathbf{r} = \partial^2\mathbf{y}/\partial v^2;$$

$$\Psi = a^{-1}(\mathbf{m}\times\mathbf{n})\cdot[(\mathbf{m}\times\mathbf{n})\times(\mathbf{p}du^2 + 2\mathbf{q}dudv + \mathbf{r}dv^2)];$$

$$y_{11} = a^{-1}(\mathbf{m}\times\mathbf{n})\cdot(\mathbf{m}\times\mathbf{n}\times\mathbf{p}), \quad y_{12} = a^{-1}(\mathbf{m}\times\mathbf{n})\cdot(\mathbf{m}\times\mathbf{n}\times\mathbf{q}),$$

$$y_{22} = a^{-1}(\mathbf{m}\times\mathbf{n})\cdot(\mathbf{m}\times\mathbf{n}\times\mathbf{r}).$$

The expansion of the products gives expressions like

$$(\mathbf{m}\times\mathbf{n})\cdot(\mathbf{m}\times\mathbf{n}\times\mathbf{p}) = \begin{vmatrix} \mathbf{m} & \mathbf{n} & \mathbf{p} \\ \mathbf{m}^2 & \mathbf{m}\cdot\mathbf{n} & \mathbf{m}\cdot\mathbf{p} \\ \mathbf{m}\cdot\mathbf{n} & \mathbf{n}^2 & \mathbf{n}\cdot\mathbf{p} \end{vmatrix};$$

such an expression represents the component of $\mathbf{p}$ perpendicular to $\mathbf{m}\times\mathbf{n}$ multiplied by the square of $\mathbf{m}\times\mathbf{n}$. The product

$$[(\mathbf{m}\times\mathbf{n})\cdot(\mathbf{m}\times\mathbf{n}\times\mathbf{p})]\cdot[(\mathbf{m}\times\mathbf{n})\cdot(\mathbf{m}\times\mathbf{n}\times\mathbf{r})] = (\mathbf{m}\times\mathbf{n})^2(\mathbf{m}\times\mathbf{n}\times\mathbf{p})\cdot(\mathbf{m}\times\mathbf{n}\times\mathbf{r}).$$

Hence

$$\Phi_S = (\mathbf{m}\times\mathbf{n}\times\mathbf{p})\cdot(\mathbf{m}\times\mathbf{n}\times\mathbf{r}) - (\mathbf{m}\times\mathbf{n}\times\mathbf{q})^2 = G\,a, \tag{106}$$

$$Ga = \begin{vmatrix} \mathbf{m}^2 & \mathbf{m}\cdot\mathbf{n} & \mathbf{m}\cdot\mathbf{r} \\ \mathbf{m}\cdot\mathbf{n} & \mathbf{n}^2 & \mathbf{n}\cdot\mathbf{r} \\ \mathbf{m}\cdot\mathbf{p} & \mathbf{n}\cdot\mathbf{p} & \mathbf{p}\cdot\mathbf{r} \end{vmatrix} - \begin{vmatrix} \mathbf{m}^2 & \mathbf{m}\cdot\mathbf{n} & \mathbf{m}\cdot\mathbf{q} \\ \mathbf{m}\cdot\mathbf{n} & \mathbf{n}^2 & \mathbf{n}\cdot\mathbf{q} \\ \mathbf{m}\cdot\mathbf{q} & \mathbf{n}\cdot\mathbf{q} & \mathbf{q}^2 \end{vmatrix}. \tag{106'}$$

If the surface is a ruled surface the form

$$\mathbf{y} = \mathbf{f}(u) + v\mathbf{g}(u)$$

is a possible parametric form. Then

$$\mathbf{m} = \mathbf{f}' + v\mathbf{g}', \quad \mathbf{n} = \mathbf{g}, \quad \mathbf{q} = \mathbf{g}', \quad \mathbf{r} = 0,$$
$$Ga = -(\mathbf{m}\times\mathbf{n}\times\mathbf{q})^2 = -(\mathbf{f}'\times\mathbf{g}\times\mathbf{g}')^2.$$

Hence: *The total curvature of any ruled surface with real rulings is negative.* If the surface is developable, i. e., if $G = 0$, we have $\mathbf{f}'\times\mathbf{g}\times\mathbf{g}' = 0$ or $\mathbf{g}' = b\mathbf{f}' + c\mathbf{g}$, where b and c are functions of u alone. Then,

$$\mathbf{M} = \frac{[\mathbf{f}' + v(b\mathbf{f}' + c\mathbf{g})]\times\mathbf{g} = (1+bv)\mathbf{f}'\times\mathbf{g}}{(1+bv)\,|\,\mathbf{f}'\times\mathbf{g}\,|} = \frac{\mathbf{f}'\times\mathbf{g}}{|\mathbf{f}'\times\mathbf{g}|}$$

is a function of the single variable u and remains constant as v changes, the tangent plane is tangent along the whole generator, and the surface is the tangent surface of a twisted curve. Hence: *All developable ruled surfaces are twisted curve surfaces.*

If the ruled surface is not developable we select as a simple canonical form that obtained by taking the directrix $\mathbf{y} = \mathbf{f}(u)$ orthogonal to the rulings and u as the arc along this curve. Then,

$$\mathbf{f}'\cdot\mathbf{f}' = 1, \quad \mathbf{f}'\cdot\mathbf{g} = 0, \quad \mathbf{g}\cdot\mathbf{g} = 1, \quad \mathbf{g}\cdot\mathbf{g}' = 0, \quad \mathbf{f}'\cdot\mathbf{f}'' = 0,$$
$$\mathbf{m}\cdot\mathbf{n} = 0, \quad \mathbf{q}\cdot\mathbf{n} = 0, \quad \mathbf{n}^2 = 1, \quad a = \mathbf{m}^2;$$

$$\mathbf{y}_{11} = a^{-1}\begin{vmatrix} \mathbf{m} & \mathbf{n} & \mathbf{p} \\ \mathbf{m}^2 & 0 & \mathbf{m}\cdot\mathbf{p} \\ 0 & 1 & \mathbf{n}\cdot\mathbf{p} \end{vmatrix} = a^{-1}(-\,\mathbf{m}\cdot\mathbf{p}\mathbf{m} + \mathbf{m}^2\mathbf{p} - \mathbf{m}^2\mathbf{n}\cdot\mathbf{p}\mathbf{n}),$$

$$\mathbf{y}_{12} = a^{-1}\begin{vmatrix} \mathbf{m} & \mathbf{n} & \mathbf{q} \\ \mathbf{m}^2 & 0 & \mathbf{m}\cdot\mathbf{q} \\ 0 & 1 & 0 \end{vmatrix} = a^{-1}(\mathbf{m}^2\mathbf{q} - \mathbf{m}\cdot\mathbf{q}\mathbf{m}),$$

$$\mathbf{y}_{22} = 0;$$

$$\Phi = \tfrac{1}{2}(\mathbf{y}_{11}\mathbf{y}_{22} + \mathbf{y}_{22}\mathbf{y}_{11}) - \mathbf{y}_{12}\mathbf{y}_{12}$$
$$= -\,a^{-2}(\mathbf{m}^2\mathbf{q} - \mathbf{m}\cdot\mathbf{q}\mathbf{m})\,(\mathbf{m}^2\mathbf{q} - \mathbf{m}\cdot\mathbf{q}\mathbf{m}).$$

Moreover, $a^{(11)} = 1/a, \quad a^{(12)} = 0, \quad a^{(22)} = 1,$

$$2\mathbf{h} = a^{-2}(-\,\mathbf{m}\cdot\mathbf{p}\mathbf{m} + \mathbf{m}^2\mathbf{p} - \mathbf{m}^2\mathbf{n}\cdot\mathbf{p}\mathbf{n}),$$
$$\Phi/a = \mathbf{h}\mathbf{h} - \boldsymbol{\mu}\boldsymbol{\mu} - \boldsymbol{\delta}\boldsymbol{\delta}, \quad \boldsymbol{\mu}\boldsymbol{\mu} + \boldsymbol{\delta}\boldsymbol{\delta} = \mathbf{h}\mathbf{h} - \Phi/a.$$
$$\boldsymbol{\mu}\boldsymbol{\mu} + \boldsymbol{\delta}\boldsymbol{\delta} = \tfrac{1}{4}a^{-4}(-\,\mathbf{m}\cdot\mathbf{p}\mathbf{m} + \mathbf{m}^2\mathbf{p} - \mathbf{m}^2\mathbf{n}\cdot\mathbf{p}\mathbf{n})\,(-\mathbf{m}\cdot\mathbf{p}\mathbf{m} + \mathbf{m}^2\mathbf{p} - \mathbf{m}^2\mathbf{n}\cdot\mathbf{p}\mathbf{n})$$
$$+\,a^{-3}(\mathbf{m}^2\mathbf{q} - \mathbf{m}\cdot\mathbf{q}\mathbf{m})\,(\mathbf{m}^2\mathbf{q} - \mathbf{m}\cdot\mathbf{q}\mathbf{m}).$$

$$\boldsymbol{\mu}\times\boldsymbol{\delta} = \frac{-1}{2\mathbf{m}^7}(-\mathbf{m}\cdot\mathbf{p}\mathbf{m} + \mathbf{m}^2\mathbf{p} - \mathbf{m}^2\mathbf{n}\cdot\mathbf{p}\mathbf{n})\times(\mathbf{m}^2\mathbf{q} - \mathbf{m}\cdot\mathbf{q}\mathbf{m}).$$

Hence $\boldsymbol{\mu}\times\boldsymbol{\delta}$ contains $\mathbf{h}$ and the indicatrix lies in a plane with the surface point, no matter how high the dimensionality of the space in which the surface lies. Hence: *A ruled surface is at each point of the four dimensional type and never of the general type*, i. e., *a ruled surface is made up of planar points*, in Levi's nomenclature.

The formulas will serve to investigate the whole surface. If we are interested only in the neighborhood of some ordinary point we may assume that the point lies on the trajectory $\mathbf{y} = \mathbf{f}(u)$, that is, $v = 0$. The formulas then simplify further; for

$$\mathbf{m} = \mathbf{f}', \quad \mathbf{p} = \mathbf{f}'', \quad \mathbf{m}\cdot\mathbf{p} = \mathbf{f}'\cdot\mathbf{f}'' = 0, \quad a = \mathbf{m}^2 = 1;$$
$$2\mathbf{h} = \mathbf{f}'' - \mathbf{g}\cdot\mathbf{f}''\mathbf{g}, \quad \Phi = -\,(\mathbf{g}' - \mathbf{f}'\cdot\mathbf{g}'\mathbf{f}')\,(\mathbf{g}' - \mathbf{f}'\cdot\mathbf{g}'\mathbf{f}').$$

Moreover,[40]

$$\pm \boldsymbol{\mu} = \tfrac{1}{2}(\mathbf{f}'' - \mathbf{g}\cdot\mathbf{f}''\mathbf{g}), \quad \mp \boldsymbol{\delta} = \mathbf{g}' - \mathbf{f}'\cdot\mathbf{g}'\mathbf{f}',$$
$$\Phi_S = G = -\,(\mathbf{g}' - \mathbf{f}'\cdot\mathbf{g}'\mathbf{f}')^2 = -\,\boldsymbol{\delta}^2.$$

It is seen from these equations that: *The mean curvature* $\mathbf{h}$ *of a ruled surface is one-half the normal component of the curvature* $\mathbf{f}''$ *of an orthogonal trajectory of the rulings; the indicatrix is a conic of which a pair of conjugate radii are the mean curvature* $\mathbf{h}$ *and the line* $\boldsymbol{\delta}$ *which is the normal component of* $\mathbf{g}'$, *which gives the rate of turning of the rulings; the total curvature of the surface is the negative of the square of the normal component of* $\mathbf{g}'$. *The ruled surface is a special type under the four-dimensional type in that the indicatrix passes through the surface point considered.* As the inverse of the pedal of an ellipse with respect to a point on the ellipse is a parabola, *the locus of points where consecutive normal planes (spaces) meet a given normal plane (space) is a parabola (parabolic cylinder) with its axis parallel to* $\mathbf{h}$.

41. Developable surfaces. One particular parametric form for a general surface,

$$x = x, \quad y = y, \quad z_i = z_i(x, y),$$

which expresses the surface as the intersection of $n - 2$ cylinders $z_i = z_i(x, y)$, is often useful. In this case the vector coördinates of the surface and the differential element of arc are

$$\boldsymbol{\rho} = x\mathbf{i} + y\mathbf{j} + \Sigma z_i\mathbf{k}_i,$$
$$d\boldsymbol{\rho} = (\mathbf{i} + \Sigma \frac{\partial z_i}{\partial x}\mathbf{k}_i)dx + (\mathbf{j} + \Sigma \frac{\partial z_i}{\partial y}\mathbf{k}_i)dy, \tag{107}$$
$$d\boldsymbol{\rho}\cdot d\boldsymbol{\rho} = \left[1 + \Sigma\left(\frac{\partial z_i}{\partial x}\right)^2\right]dx^2 + 2\Sigma\frac{\partial z_i}{\partial x}\frac{\partial z_i}{\partial y}dxdy + \left[1 + \Sigma\left(\frac{\partial z_i}{\partial y}\right)^2\right]dy^2.$$

Let p_i, q_i be the derivatives of z_i with respect to x and y. Then,

$$\mathbf{m} = \mathbf{i} + \Sigma p_i\mathbf{k}_i, \quad \mathbf{n} = \mathbf{j} + \Sigma q_i\mathbf{k}_i. \tag{107$'$}$$

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40 The actual determination of a possible set of values for $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$ may be made when the values of $\boldsymbol{\mu}\boldsymbol{\mu} + \boldsymbol{\delta}\boldsymbol{\delta}$ and $\boldsymbol{\mu}\times\boldsymbol{\delta}$ are known. In this particular case $\boldsymbol{\mu}\times\boldsymbol{\delta} = -\,\frac{1}{2}\mathbf{c}\times\mathbf{d}$ where $\mathbf{c} = \mathbf{f}'' - \mathbf{g}'\cdot\mathbf{f}''\mathbf{g}$, $\mathbf{d} = \mathbf{g}' - \mathbf{f}'\cdot\mathbf{g}'\mathbf{f}'$ and $\mathbf{h} = \frac{1}{2}\mathbf{c}$. Then since $a = 1$, $\boldsymbol{\mu}\boldsymbol{\mu} + \boldsymbol{\delta}\boldsymbol{\delta} = \mathbf{h}\mathbf{h} = \frac{1}{4}\mathbf{c}\mathbf{c} + \mathbf{d}\mathbf{d}$. If $\boldsymbol{\mu} = \mathbf{c}x$ and $\boldsymbol{\delta} = \mathbf{d}/x$, then $\boldsymbol{\mu}\boldsymbol{\mu} + \boldsymbol{\delta}\boldsymbol{\delta} = \frac{1}{4}x^2\mathbf{c}\mathbf{c} + \mathbf{d}\mathbf{d}/x^2$, and x must be unity provided $\mathbf{c}$ and $\mathbf{d}$ have distinct directions as they must have since $\boldsymbol{\mu}\times\boldsymbol{\delta} \neq 0$.

If we use r_i, s_i, t_i, for the second derivatives of z_i in accord with the usual notation, the quantities **p**, **q**, **r**, are

$$\mathbf{p} = \Sigma r_i \mathbf{k}_i, \quad \mathbf{q} = \Sigma s_i \mathbf{k}_i, \quad \mathbf{r} = \Sigma t_i \mathbf{k}_i. \tag{107''}$$

With these values, $\mathbf{y}_{11}$, $\mathbf{y}_{12}$, $\mathbf{y}_{22}$, etc. may be calculated.

We shall at this point merely calculate, from (106′),

$$\begin{aligned}
\Phi_S = Ga &= \begin{vmatrix} 1+\Sigma p_i^2 & \Sigma p_i q_i & \Sigma p_i t_i \\ \Sigma p_i q_i & 1+\Sigma q_i^2 & \Sigma q_i t_i \\ \Sigma p_i r_i & \Sigma q_i r_i & \Sigma r_i t_i \end{vmatrix} \\
&\quad - \begin{vmatrix} 1+\Sigma p_i^2 & \Sigma p_i q_i & \Sigma p_i s_i \\ \Sigma p_i q_i & 1+\Sigma q_i^2 & \Sigma q_i s_i \\ \Sigma p_i s_i & \Sigma q_i s_i & \Sigma s_i^2 \end{vmatrix} \\
&= \Sigma(r_i t_i - s_i^2)\left[\begin{vmatrix} 1+\Sigma p_i^2 & \Sigma p_i q_i \\ \Sigma p_i q_i & 1+\Sigma q_i^2 \end{vmatrix} - p_i^2(1+\Sigma q_i^2)\right. \\
&\quad \left. - q_i^2(1+\Sigma p_i^2) + 2p_i q_i \Sigma p_i q_i\right] \qquad (108) \\
&\quad + \Sigma_{ij}(t_i r_j - s_i s_j)\,[-p_i p_j(1+\Sigma q_i^2) + p_i q_j \Sigma p_i q_i \\
&\quad\quad + q_i p_j \Sigma p_i q_i - q_i q_j(1+\Sigma p_i^2)].
\end{aligned}$$

In the particular case $n = 4$ where i and j run over the indices 1 and 2, the formula becomes,

$$\begin{aligned}
Ga = (r_1 t_1 - s_1^2)\,(1 + p_2^2 + q_2^2) + (r_2 t_2 - s_2^2)\,(1 + p_1^2 + q_1^2) \\
- (t_1 r_2 + r_2 t_1 - 2 s_1 s_2)\,(p_1 p_2 + q_1 q_2).
\end{aligned} \tag{108'}$$

The case $n > 4$ is much more complicated but consists of a sum of terms $rt - s^2$, with coefficients, and some supplementary terms.

If the surface is a twisted curve surface its rulings will project into lines and hence each of its projections z_i must be a twisted curve surface and the terms $rt - s^2$ vanish; but as $G = 0$ there are supplementary conditions to be satisfied, the condition in case $n = 4$ being

$$(t_1 r_2 + r_2 t_1 - 2 s_1 s_2)\,(p_1 p_2 + q_1 q_2) = 0.$$

But the surface may be developable, that is, $G = 0$, without making the individual terms $rt - s^2$ vanish. Indeed if in four dimensions

we assume the projection $z_1 = z_1(x, y)$ at random, (108′) equated to zero becomes a partial differential equation of the second order for the other projection $z_2 = z_2(x,y)$, and any solution z_2 of this equation, taken with z_1, will define a developable surface in four dimensions. In case $n > 4$ we may assume at random $n - 3$ projections $z_1 = z_1(x, y)$, $z_2 = z_2(x, y), \ldots \ldots, z_{n-3} = z_{n-3}(x, y)$, and proceed to solve the differential equation obtained by setting (108) equal to zero for the projection z_{n-2} which taken with the assumed $n - 3$ projections, will determine a developable. *In more than three dimensions developable 2-surfaces therefore are either* 1°, *ruled developables which are twisted curve surfaces, or* 2°, *non-ruled developable surfaces.*

As a particularly simple case of a non-ruled developable for $n = 4$ we may take

$$z_1 = \tfrac{1}{2}(x^2 + y^2), \qquad z_2 = xy.$$

This surface satisfies (108′) but the individual terms $rt - s^2$ do not vanish. If we turn the axes of z_1 and z_2 and of x and y through 45° in their respective planes and change the scale, the surface may take the form

$$z_1 = \tfrac{1}{2}x^2, \qquad z_2 = \tfrac{1}{2}y^2.$$

In this case each of the surfaces $z_1 = \frac{1}{2}x^2$ and $z_2 = \frac{1}{2}y^2$ taken as a three dimensional surface is developable. But the four dimensional surface is not a ruled surface. In other words the projections of a non-ruled developable may each be ruled developables. All surfaces of the type

$$z_1 = z_1(x), \qquad z_2 = z_2(y)$$

are developable, because the element of arc is

$$(1 + z_1'^2)dx^2 + (1 + z_2'^2)dy^2 = dX^2 + dY^2,$$
$$dX = \sqrt{1 + z_1'^2}\, dx, \quad dY = \sqrt{1 + z_2'^2}\, dy.$$

Such surfaces, however, are not in general ruled.

42. Development of a surface about a point. There is a great simplification in our formulas if we restrict ourselves to the neighborhood of a single point of the surface and take the tangent plane at that point as the xy-plane. (This is the method followed at length by Kommerell in the four dimensional case.) In general we have for the surface,

$$z_i = \tfrac{1}{2}(A_i x^2 + 2B_i xy + C_i y^2), \qquad i = 1, 2, \ldots n - 2,$$

up to infinitesimals of the third order. The $3n - 6$ constants are not geometrically independent because of the arbitrary choice of the direction of the axes in the xy-plane. There are only $3n - 7$ independent constants. There are $3(n-2) - 6 = 3n - 12$ degrees of freedom for a plane in S_{n-3} and five degrees of freedom for an ellipse in the plane. The count of constants indicates, therefore, that the indicatrix may be any ellipse in the normal space. We may examine this proposition critically by reference to the general formulas (107), (107′), (107″).

For this case $\mathbf{m} = \mathbf{i}$, $\mathbf{n} = \mathbf{j}$, $a_{11} = 1$, $a_{12} = 0$, $a_{22} = 1$, $a = 1$.

$$a^{(11)} = 1, \quad a^{(12)} = 0, \quad a^{(22)} = 1, \quad \mathbf{p} = \Sigma A_i \mathbf{k}_i, \quad \mathbf{q} = \Sigma B_i \mathbf{k}_i,$$
$$\mathbf{r} = \Sigma C_i \mathbf{k}_i, \quad \mathbf{y}_{11} = \mathbf{p}, \quad \mathbf{y}_{12} = \mathbf{q}, \quad \mathbf{y}_{22} = \mathbf{r}, \quad 2\mathbf{h} = \Sigma(A_i + C_i)\mathbf{k}_i.$$

The center of the indicatrix may therefore be any point, and is the same point for any two surfaces for which $A_i + C_i$ are the same, $i = 1, 2, \ldots n-2$. The plane $\boldsymbol{\mu}\times\boldsymbol{\delta}$ is determined by (102) as

$$2\boldsymbol{\mu}\times\boldsymbol{\delta} = \mathbf{y}_{12}\times(\mathbf{y}_{22} - \mathbf{y}_{11}) = \Sigma B_i \mathbf{k}_i \times \Sigma(C_j - A_j)\mathbf{k}_j.$$

As $A_j + C_j$ and $A_j - C_j$ are independent, $\boldsymbol{\mu}\times\boldsymbol{\delta}$ may be any plane in the normal S_{n-2}. The work may now be simplified by choosing $\mathbf{h}$ as the axis z_1 and by taking the axes z_2, z_3 in the space $\mathbf{h}\times\boldsymbol{\mu}\times\boldsymbol{\delta}$. The equations of the surface reduce to

$$z_1 = \tfrac{1}{2}[A_1x^2 + 2B_1xy + C_1y^2], \qquad z_2 = \tfrac{1}{2}[A_2(x^2 - y^2) + 2B_2xy],$$
$$z_3 = \tfrac{1}{2}[A_3(x^2 - y^2) + 2B_3xy], \qquad z_i = 0, \qquad i = 4, 5, \ldots n-2.$$

That z_2 and z_3 take these special forms is due to the fact (§38) that the mean curvature of each must vanish. A proper orientation of the xy axes makes $B_1 = 0$. By properly choosing the axes $\mathbf{k}_2$, $\mathbf{k}_3$ we may make B_2 vanish. We have then as a canonical form for the surface,

$$z_1 = \tfrac{1}{2}[A_1x^2 + C_1y^2], \quad z_2 = \tfrac{1}{2}A_2(x^2 - y^2), \tag{109}$$
$$z_3 = \tfrac{1}{2}[A_3(x^2 - y^2) + 2B_3xy], \quad z_i = 0, \quad i > 3.$$

Now, $2\mathbf{h} = (A_1 + C_1)\mathbf{k}_1$, $\quad 2\boldsymbol{\mu}\times\boldsymbol{\delta} = B_3\mathbf{k}_3\times[(C_1 - A_1)\mathbf{k}_1 - 2A_2\mathbf{k}_2]$.

If we set $\quad A_1 = h + e, \quad C_1 = h - e, \quad A_2 = f, \quad A_3 = A, \quad B_3 = B,$

$$z_1 = \tfrac{1}{2}[h(x^2 + y^2) + e(x^2 - y^2)], \quad z_2 = \tfrac{1}{2}f(x^2 - y^2),$$
$$z_3 = \tfrac{1}{2}[A(x^2 - y^2) + 2Bxy], \quad z_i = 0, \quad i > 3. \tag{109′}$$

This is a very useful standard form for the expansion of a surface near a given point. Then $\boldsymbol{\mu}\times\boldsymbol{\delta} = B(e\mathbf{k}_1 + f\mathbf{k}_2)\times\mathbf{k}_3$, and only the ratio $e{:}f$ is effective in changing the plane $\boldsymbol{\mu}\times\boldsymbol{\delta}$. The equation therefore contains three constants after $\mathbf{h}$ and the plane $\boldsymbol{\mu}\times\boldsymbol{\delta}$ are satisfied, namely A, B, and e or f, which may suffice to determine the indicatrix with its center and plane already fixed.

Using polar coordinates (ρ, θ) in the tangent plane, we have

$$z_1 = \tfrac{1}{2}\rho_2(h + e\cos 2\theta), \quad z_2 = \tfrac{1}{2}\rho^2 f\cos 2\theta, \tag{109''}$$
$$z_3 = \tfrac{1}{2}\rho^2(A\cos 2\theta + B\sin 2\theta), \quad z_i = 0, \quad i > 3.$$

The normal vector distance, of the surface curve in the direction θ, above the tangent plane is therefore

$$\tfrac{1}{2}\rho^2[(h + e\cos 2\theta)\mathbf{k}_1 + f\cos 2\theta\mathbf{k}_2 + (A\cos 2\theta + B\, 2\sin\theta)\mathbf{k}_3], \tag{110}$$

and the normal curvature $\boldsymbol{\alpha}$ is

$$\boldsymbol{\alpha} = (h + \mathrm{e}\cos 2\theta)\mathbf{k}_1 + f\cos 2\theta\mathbf{k}_2 + (A\cos 2\theta + B\sin 2\theta)\mathbf{k}_3. \tag{111}$$

The vectors $\boldsymbol{\delta} = \boldsymbol{\alpha} - \mathbf{h}$ and $\boldsymbol{\mu}$, which is $\boldsymbol{\delta}$ advanced 45° in θ, are

$$\boldsymbol{\delta} = (e\mathbf{k}_1 + f\mathbf{k}_2 + A\mathbf{k}_3)\cos 2\theta + B\mathbf{k}_3\sin 2\theta,$$
$$\boldsymbol{\mu} = -(e\mathbf{k}_1 + f\mathbf{k}_2 + A\mathbf{k}_3)\sin 2\theta + B\mathbf{k}_3\cos 2\theta. \tag{112}$$

The indicatrix reduces to a line when and only when $B = 0$ or $e = f = 0$. The former may be regarded as the general case. It appears then that $\boldsymbol{\delta}$ may describe any line in the normal S_3 and the range of $\boldsymbol{\delta}$ may be for any distance $(e^2 + f^2 + A^2)^{\frac{1}{2}}$ along that line. If the indicatrix does not reduce to a line, and if u, v denote coordinates referred to the unit orthogonal vectors $\mathbf{k}_3$ and $\mathbf{k}' = (e\mathbf{k}_1 + f\mathbf{k}_2)/(e^2 + f^2)^{\frac{1}{2}}$, we have

$$u = A\cos 2\theta + B\sin 2\theta, \quad v = (e^2 + f^2)^{\frac{1}{2}}\cos 2\theta.$$

Let $e = \lambda\cos\varphi$, $f = \lambda\sin\varphi$. The plane $\boldsymbol{\mu}\times\boldsymbol{\delta}$ determines φ but not λ. The equation of the indicatrix in its plane is then

$$\lambda^2 u^2 - 2\lambda A uv + (A^2 + B^2)v^2 = B^2\lambda^2. \tag{113}$$

Any ellipse may be written as

$$au^2 + 2buv + cv^2 = 1, \quad a > 0, \quad c > 0, \quad ac - b^2 > 0.$$

To determine the outstanding constants λ, A, B, so that the indicatrix takes this form we have merely to take

$$B^2 = \frac{1}{a}, \quad A^2 = \frac{b^2}{a(ac - b^2)}, \quad \lambda^2 = \frac{a}{ac - b^2},$$

and this choice is always possible. Hence we have shown that: *The indicatrix may be any ellipse or any segment of a straight line in the normal S_{n-2}.* (We are ordinarily more interested in the domain of reals than in the domain of complex numbers, and this theorem holds for reals.)

If we are working in the special case of four dimensions we have merely to set $f = 0$ throughout the work. The results are the same for the special case as for the general case,— the indicatrix may be any ellipse or segment of a line in the normal plane.

The dyadic $\Phi = (\mathbf{hh} - \boldsymbol{\mu\mu} - \boldsymbol{\delta\delta})a = \frac{1}{2}\mathbf{y}_{11}\mathbf{y}_{22} + \frac{1}{2}\mathbf{y}_{22}\mathbf{y}_{11} - \mathbf{y}_{12}\mathbf{y}_{21}$, is

$$\begin{aligned} \Phi = (h^2 - e^2)\mathbf{k}_1\mathbf{k}_1 - f^2\mathbf{k}_2\mathbf{k}_2 - (A^2 + B^2)\mathbf{k}_3\mathbf{k}_3 \\ - fA(\mathbf{k}_2\mathbf{k}_3 + \mathbf{k}_3\mathbf{k}_2) - Ae(\mathbf{k}_1\mathbf{k}_3 + \mathbf{k}_3\mathbf{k}_1) - fe(\mathbf{k}_1\mathbf{k}_2 + \mathbf{k}_2\mathbf{k}_1), \quad (114) \\ \Phi_S = G = h^2 - e^2 - f^2 - A^2 - B^2. \end{aligned}$$

The vertex of Cone II is located at

$$\boldsymbol{\nu} = \frac{(\boldsymbol{\mu}\times\boldsymbol{\delta})\cdot(\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h})}{(\boldsymbol{\mu}\times\boldsymbol{\delta}\times\mathbf{h})^2} = \frac{B}{B}\frac{(f\mathbf{k}_1 - e\mathbf{k}_2)}{fh}, \tag{115}$$

and retreats to infinity if h vanishes unless special conditions are fulfilled. If $B = 0$ the vertex is indeterminate. The determinant of Φ reduces to $f^2h^2B^2$ and hence Φ becomes singular and Cone II degenerates when $f = 0$ or $h = 0$ or $B = 0$. It is however clear, from the equations (109′) of the surface, that if $fBh = 0$, the surface lies in four dimensions at the point considered. Hence for a true five dimensional surface in the neighborhood of a point, the indicatrix must be a true ellipse (cannot degenerate to a linear segment) in a plane not containing $\mathbf{h}$, and Cone II cannot degenerate nor its vertex retreat to infinity.

Special cases. It remains to discuss the case for a locally four dimensional surface

$$z_1 = \tfrac{1}{2}[h(x^2 + y^2) + e(x^2 - y^2)], \quad z_3 = \tfrac{1}{2}[A(x^2 - y^2) + 2Bxy].$$

Here $\boldsymbol{\mu}\times\boldsymbol{\delta} = Be\mathbf{k}_1\times\mathbf{k}_3$. This vanishes only when $B = 0$ or $e = 0$. As

$\boldsymbol{\delta} = (e\mathbf{k}_1 + A\mathbf{k}_3)\cos 2\theta + B\mathbf{k}_3\sin 2\theta$, we see that $B = 0$ represents the general case. If $Be \neq 0$, the indicatrix (Conic I) is a true ellipse with central radius $\boldsymbol{\delta}$. Referred to its center the equation of the indicatrix is

$$e^2z_3^2 - 2Aez_1z_3 + (A^2 + B^2)z_1^2 = B^2e^2,$$

as may be seen from (113). To find the locus of the intersection of consecutive normal planes we need the inverse of the pedal of the ellipse with respect to the origin. One observation may be made in advance: the conic (Conic II) which will be found must contain the origin in its interior.

The calculation of the inverse pedal may be carried through neatly by vectors. If Ω be the selfconjugate two dimensional dyadic that gives the conic referred to its center as $\boldsymbol{\delta}\cdot\Omega\cdot\boldsymbol{\delta} = 1$, $\Omega\cdot\boldsymbol{\delta}$ is the normal, and the equation of the tangent at $\boldsymbol{\delta}$, or at $\boldsymbol{\delta} + \mathbf{h}$ when referred to the origin, is

$$(\mathbf{r} - \mathbf{h} - \boldsymbol{\delta})\cdot\Omega\cdot\boldsymbol{\delta} = 0 \quad \text{or} \quad \mathbf{r}\cdot\Omega\cdot\boldsymbol{\delta} = 1 + \mathbf{h}\cdot\Omega\cdot\boldsymbol{\delta},$$

where $\mathbf{r}$ is the radius vector. Hence

$$\frac{\mathbf{r}\cdot\Omega\cdot\boldsymbol{\delta}}{1 + \mathbf{h}\cdot\Omega\cdot\boldsymbol{\delta}} = 1, \quad \text{and} \quad \mathbf{p} = \frac{\Omega\cdot\boldsymbol{\delta}}{1 + \mathbf{h}\cdot\Omega\cdot\boldsymbol{\delta}}$$

is the radius vector of the inverse of the pedal. Then

$$\boldsymbol{\delta} = \frac{\Omega^{-1}\cdot\mathbf{p}}{1 - \mathbf{h}\cdot\mathbf{p}} \quad \text{and} \quad \frac{\mathbf{p}\cdot\Omega^{-1}\cdot\mathbf{p}}{(1 - \mathbf{h}\cdot\mathbf{p})^2} = 1.$$

The inverse of the pedal is therefore,

$$\mathbf{p}\cdot\Omega^{-1}\cdot\mathbf{p} = 1 - 2\mathbf{h}\cdot\mathbf{p} + (\mathbf{h}\cdot\mathbf{p})^2.$$

Taking Ω from (114) with $f = 0$ we find Ω^{-1} at once and hence the desired locus (Conic II) is

$$(e^2 - h^2)z_1^2 + 2Aez_1z_3 + (A^2 + B^2)z_3^2 + 2hz_1 = 1. \tag{116}$$

The discriminant of the first three terms is $h^2(A^2 + B^2) - B^2e^2$. The conic is an ellipse, parabola, or hyperbola according as this expression

is negative, zero, or positive.[41] The conic breaks up into two lines if $Be = 0$, that is if the indicatrix is a linear segment.

The degenerate case $B = 0$ requires a little more investigation to find what happens to consecutive normal spaces. If we observe what happens as we pass from a true ellipse to a segment, we see that the points of intersection of consecutive normal planes bunch themselves more and more closely about the point $z_1 = 1/h$, $z_3 = -e/Ah$, which is the inverse of the foot of the perpendicular to the segment from the surface point. It therefore appears that the normal planes all pass through a common point $(0, 0, 1/h, -e/Ah)$ in this special case; the two nearby planes in the direction of the axes x, y may be said to cut the normal plane in the lines $z_1\ (h + e) + Az_3 = 1$ and $z_1(h + e) - Az_3 = 1$ respectively. These lines are those into which (116) factors and are perpendicular to the lines which join the surface point to the extremities of the indicatrix.

There is a special case under the case $B = 0$, namely that in which the indicatrix, now a segment, is collinear with $\mathbf{h}$. The surface is then three dimensional in the neighborhood of the point, or the point is axial in Levi's nomenclature. The common intersection of the consecutive normal planes has retreated to infinity and the locus reduces to two parallel straight lines which are the intersection of the consecutive normal planes in the x and y directions with the normal at the given point — thus consecutive normal planes do not in general meet that normal plane.

43. Segre's Characteristics. The points for which $B = 0$, that is, those where the indicatrix reduces to a linear segment have one property of importance in common with surfaces in three dimensions. For *if the indicatrix reduces to a linear segment, there are two directions on the surface, namely those corresponding to the ends of the linear segment, for which* $\boldsymbol{\mu} = 0$, *and these are orthogonal directions, and for them the normal curvature is a maximum or a minimum. If these lines be taken as parametric curves the second fundamental (vector) form and the first fundamental form reduce simultaneously to the sum of squares,*

$$\varphi = a_{11}dx_1^2 + a_{22}dx_2^2, \quad \Psi = \mathbf{y}_{11}dx_1^2 + \mathbf{y}_{22}dx_2^2.$$

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41 Kommerell distinguishes these cases by saying that the surface point is elliptic, hyperbolic, or parabolic, but though this distinction may be useful in the case of surfaces lying in a 4-space and possibly at planar points in general, there is apparently no similar classification in general surface theory.

Further: the rate of change of the tangent plane squared is, from (73), (80), (82),

$$\frac{d\mathbf{M}}{ds} \times \frac{d\mathbf{M}}{ds} = (\boldsymbol{\alpha}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\mu}) \times (\boldsymbol{\alpha}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\mu}) = 2\boldsymbol{\xi}\times\boldsymbol{\eta}\times\boldsymbol{\mu}\times\boldsymbol{\alpha}. \quad (117)$$

As $\boldsymbol{\xi}\times\boldsymbol{\eta}$ and $\boldsymbol{\mu}\times\boldsymbol{\alpha}$ are completely perpendicular, the only possibility for the product to vanish is that $\boldsymbol{\mu}\times\boldsymbol{\alpha} = 0$. This will vanish when $\boldsymbol{\mu} = 0$ and hence: *If the indicatrix is a linear segment there are two directions for which the rate of change* $d\mathbf{M}/ds$ *is a simple plane vector.* In this case $\mathbf{M}$ and $d\mathbf{M}$ intersect in a line. *There are then only two directions in which consecutive tangent planes intersect in a line.*

If the indicatrix does not reduce to a linear segment the only way that $\boldsymbol{\mu}\times\boldsymbol{\alpha}$ can vanish is to have $\boldsymbol{\mu}$ and $\boldsymbol{\alpha}$ parallel or $\boldsymbol{\alpha}$ vanish. Now the latter alternative will happen when and only when the indicatrix (now an actual ellipse) passes through the surface point and in this case there is only a single direction in which $d\mathbf{M}$ is a simple vector. If $\mathbf{h}\times\boldsymbol{\mu}\times\boldsymbol{\delta} = 0$, that is, *if the surface at the point is four dimensional* (i. e., *planar*), *there are two directions on the surface for which* $\boldsymbol{\mu}\times\boldsymbol{\alpha} = 0$, *namely, those which make* $\boldsymbol{\alpha}$ *tangent to the indicatrix, for these directions, and only for these,* $d\mathbf{M}$ *is a simple plane and consecutive tangent planes intersect in a line.* These two directions cannot be perpendicular and may be imaginary, they are coincident in the case where $\boldsymbol{\alpha}$ may vanish. *If the surface at the point is five dimensional,* $d\mathbf{M}$ *is never a simple plane.*

It appears therefore that in no case above three dimensions can conjugate directions be defined by considering intersections of consecutive tangent planes.

We may express upon the $\mathbf{y}$'s the condition of degeneracy. First if the surface is four dimensional at the point, then at that point $\mathbf{y}_{11}$, $\mathbf{y}_{12}$, $\mathbf{y}_{22}$ must be coplanar and a linear relation

$$A\mathbf{y}_{11} + B\mathbf{y}_{12} + C\mathbf{y}_{22} = 0 \quad (118)$$

must subsist between these. Next if the ellipse collapses into a segment, the condition (102) for $\boldsymbol{\mu}\times\boldsymbol{\delta} = 0$ may be used to show that the normals, $\mathbf{y}_{11}/a_{11}$, $\mathbf{y}_{12}/a_{12}$, $\mathbf{y}_{22}/a_{22}$ are termino-collinear and the relation

$$Aa_{11} + Ba_{12} + Ca_{22} = 0$$

subsists between the coefficients A, B, C in (118). Finally if the

linear segment is collinear with $\mathbf{h}$, the normals are all collinear and must satisfy (118) and an additional relation

$$A'\mathbf{y}_{11} + B'\mathbf{y}_{12} + C'\mathbf{y}_{22} = 0. \tag{118'}$$

Segre [42] showed that if the coordinates of a surface satisfy the relation (118) at each point, there is traced on the surface a double system of curves, called characteristics, having the property that tangent planes to the surface in two infinitely near points in the direction of one of these characteristics will intersect in a line tangent to the other. Also an S_{n-1} passing through the tangent plane will cut the surface in a curve having a node at the point of contact and such that the tangents at the node are separated harmonically by the tangents to the characteristics. The direction of the nodal tangents correspond to the points in which a line drawn through the surface point cuts the indicatrix. The considerations above given show that these surfaces are those for which $\mathbf{h}\times\boldsymbol{\mu}\times\boldsymbol{\delta} = 0$ (see §39).

If the equation (118) is of the parabolic type, that is if $B^2 - 4AC = 0$, the two characteristics will coincide. If this happens Moore [43] showed that the characteristics have the property that their tangents have three point contact with the surface. For this type the indicatrix passes through the surface point and consequently one of the nodal tangents always coincides with the tangent to the (double) characteristic.

Segre also showed that a surface whose coordinates satisfy two equations (118), (118′) either lies in a three space or else consist of the tangents to a twisted curve. These are the surfaces for which our invariant $(\boldsymbol{\mu}\times\boldsymbol{\delta})^2 - \Phi_{2S}/a^2$ of §39 vanishes, and the statement there made is thus substantiated.

If the indicatrix degenerates into a linear segment not passing through the surface point, the two characteristics are perpendicular, and this is the only case in which the characteristics are at right angles. If the linear segment passes through the surface point two cases arise. 1° If one end of the segment lies on the surface then at that point the surface has the character of a twisted curve surface. If the condition is satisfied identically the surface is a twisted curve surface. 2° If the segment does not have an end in the surface then

42 Segre, Su una classa di superficie degl'iperspazi, ecc., *Atti di Torino*, 1907.
43 C. L. E. Moore, Surfaces in hyperspace which have a tangent line with three point contact passing through each point, *Bull. Amer. Math. Soc.*, **18**, 1912.

at the point the surface has the character of a three dimensional surface which is not a developable. If this condition is satisfied identically the surface must lie in three dimensions.

As an application of these results we may show that a minimum ruled surface must always lie in three dimensions and consequently be the helicoid. For as the surface is ruled the indicatrix reduces to a segment which, as $\mathbf{h} = 0$, must pass through the surface point and indeed have that point for mid point and hence the surface must lie in three dimensions.

44. Principal directions. If we take the value of $\boldsymbol{\alpha}'$ from (89), we find, as the condition that $\boldsymbol{\alpha}'$ shall be maximum or minimum in magnitude.

$$\begin{aligned} 0 = \boldsymbol{\alpha}'\cdot d\boldsymbol{\alpha}' &= (\mathbf{h} + \boldsymbol{\mu}\sin 2\theta + \boldsymbol{\delta}\cos 2\theta)\cdot(\boldsymbol{\mu}\cos 2\theta - \boldsymbol{\delta}\sin 2\theta) = \boldsymbol{\alpha}'\cdot\boldsymbol{\mu}' \\ &= \mathbf{h}\cdot\boldsymbol{\mu}\cos 2\theta - \mathbf{h}\cdot\boldsymbol{\delta}\sin 2\theta + (\boldsymbol{\mu}^2 - \boldsymbol{\delta}^2)\sin 2\theta\cos 2\theta \\ &\qquad + (\boldsymbol{\mu}\cdot\boldsymbol{\delta})(\cos^2 2\theta - \sin^2 2\theta). \end{aligned}$$

If we let $x = \tan\theta$, the resulting equation in x is

$$\begin{aligned} x^4(\boldsymbol{\mu}\cdot\boldsymbol{\delta} - \boldsymbol{\mu}\cdot\mathbf{h}) + 2x^3(\boldsymbol{\delta}^2 - \boldsymbol{\mu}^2 - \mathbf{h}\cdot\boldsymbol{\delta}) - 6x^2\boldsymbol{\mu}\cdot\boldsymbol{\delta} + 2x(\boldsymbol{\mu}^2 - \boldsymbol{\delta}^2 - \mathbf{h}\cdot\boldsymbol{\delta}) \\ + \boldsymbol{\mu}\cdot(\boldsymbol{\delta}\times\mathbf{h}) = 0. \end{aligned}$$

This is of the fourth degree and hence there are four directions of maximum or minimum for the magnitude of the normal curvature (Kommerell). Two of these directions must be real; for if we choose $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$, which may be any radii of the indicatrix, as the axes of the indicatrix, $\boldsymbol{\mu}\cdot\boldsymbol{\delta} = 0$, and the coefficients of the first and last terms are opposite in sign. If a single pair of these four directions are orthogonal, it must be possible to choose $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$ so that $x = 0$ and $x = \infty$ satisfy the equations, that is, so that $\boldsymbol{\mu}\cdot\boldsymbol{\delta} \pm \boldsymbol{\mu}\cdot\mathbf{h} = 0$ or $\boldsymbol{\mu}\cdot\boldsymbol{\delta} = \boldsymbol{\mu}\cdot\mathbf{h}$. This means that: *If two of the directions of maximum or minimum normal curvature are perpendicular one of the axes of the indicatrix must be perpendicular to* $\mathbf{h}$. If the four directions are perpendicular in pairs x and $-1/x$ must satisfy the biquadratic and

$$\begin{aligned} \boldsymbol{\mu}\cdot\boldsymbol{\delta} - \boldsymbol{\mu}\cdot\mathbf{h} = \boldsymbol{\mu}\cdot\boldsymbol{\delta} + \boldsymbol{\mu}\cdot\mathbf{h},\quad \boldsymbol{\mu}^2 - \boldsymbol{\delta}^2 - \mathbf{h}\cdot\boldsymbol{\delta} = \boldsymbol{\mu}^2 - \boldsymbol{\delta}^2 + \mathbf{h}\cdot\boldsymbol{\delta} \quad \text{or} \\ \boldsymbol{\mu}\cdot\mathbf{h} = \mathbf{h}\cdot\boldsymbol{\delta} = 0. \end{aligned}$$

Hence: *If the directions of maximum or minimum normal curvature are perpendicular in pairs, the plane of the indicatrix must be perpendicular to* $\mathbf{h}$, *that is,* $\mathbf{h}$ *must be along the axis of Cone* I.

Definition 1. Kommerell calls the directions, which give a maximum or minimum magnitude to the normal curvature, principal directions and points out that for the principal directions (in the four dimensional case) the point of intersection of consecutive normal planes lies in the osculating plane of the normal section through that direction. This is clear from the configuration of the indicatrix when $\boldsymbol{\alpha}\cdot\boldsymbol{\mu} = 0$. For here the tangent PT at the point P of the indicatrix is perpendicular to OP the vector from the surface point O to P, and consequently P' the inverse of the pedal of P, lies on OP. [He calls the conic which is the inverse of the pedal of the indicatrix (our Conic I) the characteristic (our Conic II) and shows that the principal directions correspond to the lines OP' which may be drawn from O perpendicular to the characteristic. These lines are the same as those perpendicular to the indicatrix.] In this way he generalizes one property of the principal directions in three dimensions. The generalization is far from perfect. For in the case of three dimensions consecutive normals do not intersect in general, but do intersect for principal directions, and the intersection lies in the osculating plane of the normal section through a principal direction. In the four dimensional case the normal planes in general intersect, but except for the principal directions the point of intersection does not lie in the normal $\boldsymbol{\alpha}$, and is not in the osculating plane of the normal section through the direction.

The result may be generalized to the general case of a surface in five (or more) dimensions. For if O be the surface point, F the foot of the perpendicular upon the plane of the indicatrix, and P a point such that the tangent PT is perpendicular to OP, then as PT is perpendicular to OF, PT is perpendicular to the plane OFP. Hence FP is perpendicular to PT. Thus the points P of the indicatrix which correspond to Kommerell's principal directions on the surface are those for which the radius FP from the foot of the perpendicular OF is perpendicular to the indicatrix. Moreover, if F' be the inverse of F all the normal three spaces pass through F'. Consecutive normal spaces intersect in a line through F' perpendicular to the plane of $\boldsymbol{\alpha} = OP$ and $\boldsymbol{\mu} = PT$. Hence the intersection $F'P'$ of consecutive normal spaces cuts the line OP in some point P'. Thus: *one of the* ∞^1 *points of intersection of consecutive normal spaces lies in the osculating plane of the normal section in case that section corresponds to a direction of maximum or minimum normal curvature.*

As Kommerell points out, in the special case of a surface which at O is of the three dimensional type, the condition $\boldsymbol{\alpha}\cdot\boldsymbol{\mu} = 0$, breaks down

into $\boldsymbol{\alpha} = 0$ and $\boldsymbol{\mu} = 0$. The principal directions corresponding to $\boldsymbol{\mu} = 0$ become the true principal directions while those corresponding to $\boldsymbol{\alpha} = 0$ become the asymptotic directions. In the four dimensional case when at O the indicatrix reduces to a linear segment (not passing through O) the condition $\boldsymbol{\alpha}\cdot\boldsymbol{\mu} = 0$ breaks up into $\boldsymbol{\mu} = 0$ and $\boldsymbol{\alpha}\cdot\boldsymbol{\mu} = 0$ with $\boldsymbol{\mu} \neq 0$. The directions for which $\boldsymbol{\mu} = 0$ correspond to the extremities of the segment, which may be called the true principal directions and are perpendicular, while the directions for which $\boldsymbol{\alpha}\cdot\boldsymbol{\mu} = 0$ ($\boldsymbol{\mu} \neq 0$) correspond to the directions which may be called asymptotic from analogy. These directions may be real coincident or imaginary, but in any case are bisected by the principal directions, since the two coincident points in which a line cuts the linear segment correspond to values of θ respectively greater than and less than those for which $\boldsymbol{\mu} = 0$ by equal amounts.

Definition 2. There are other ways of generalizing the principal directions to higher dimensions. For ordinary principal directions $\boldsymbol{\mu} = 0$, that is, $\boldsymbol{\mu}^2$ has a minimum. The lines for which $\boldsymbol{\mu}^2$ is a maximum corresponds to the bisectors of the principal directions. If we desire we can define as principal directions those for which $\boldsymbol{\mu}^2$ is a minimum or maximum (it is not important to distinguish between the two extremes when the indicatrix does not degenerate). Then the principal directions on the surface would be four in number, spaced equally at angles of 45° around the point on the surface and corresponding to the axes of the indicatrix.

Definition 3. There is another property which will define lines of curvature in ordinary space on all but minimal surfaces. If any direction λ be drawn on the surface at a point, the change of the normal $d\mathbf{n}$ along that line has a definite direction. It is possible to find another direction λ' such that the change $d\mathbf{n}'$ of the normal along that direction is perpendicular to $d\mathbf{n}$. In general λ' is not perpendicular to λ. But for the principal directions λ' and λ are perpendicular. Thus: *The principal directions are the pair of perpendicular directions for which the differential changes of the normal are also perpendicular.* We shall examine the value of this (third) definition for principal directions in any number of dimensions. We may consider the equation

$$\frac{d\mathbf{M}}{ds_1}\cdot\frac{d\mathbf{M}}{ds_2} = 0, \tag{119}$$

which will connect two directions on the surface. Instead of setting up the relation in general we shall use formulas (73), (80), (82), (84)

to express the condition that, for two perpendicular directions, the differential planes are orthogonal. We find

$$\frac{d\mathbf{M}}{ds}\cdot\frac{d\mathbf{M}}{d\bar{s}} = (\boldsymbol{\alpha}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\mu})\cdot(\boldsymbol{\mu}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\beta}) = \boldsymbol{\alpha}\cdot\boldsymbol{\mu} + \boldsymbol{\beta}\cdot\boldsymbol{\mu} = 2\mathbf{h}\cdot\boldsymbol{\mu} = 0. \tag{120}$$

The condition for principal directions is therefore now $\mathbf{h}\cdot\boldsymbol{\mu} = 0$; the directions on the surface are those for which $\boldsymbol{\mu}$ is perpendicular to $\mathbf{h}$. There is one line in the plane of the indicatrix that satisfies this condition on $\boldsymbol{\mu}$, namely the intersection of the plane of the indicatrix with the plane through the end of $\mathbf{h}$ perpendicular to $\mathbf{h}$. Two perpendicular directions on the surface are determined by the two opposite values of $\boldsymbol{\mu}$. Hence: *By definition 3 there are just two principal directions through each point of the surface, and these are orthogonal.* On a surface for which $\boldsymbol{\mu} = 0$ these two directions coincide with those previously called principal. In case $\mathbf{h} = 0$, the condition is satisfied for any direction on the surface, and in case $\mathbf{h}$ is not zero but is along the axis of Cone I, the condition is also satisfied identically. This last case may perhaps be likened to an umbilic in ordinary surface theory — for at an umbilic the principal directions are indeterminate.

The expression $d\mathbf{M}/ds = \boldsymbol{\alpha}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\mu}$ gives

$$(d\mathbf{M}/ds)^2 = \boldsymbol{\alpha}^2 + \boldsymbol{\mu}^2 = \mathbf{h}^2 + \boldsymbol{\delta}^2 + \boldsymbol{\mu}^2 + 2\mathbf{h}\cdot\boldsymbol{\delta}.$$

As the expressions $\mathbf{h}^2$ and $\boldsymbol{\mu}^2 + \boldsymbol{\delta}^2$ are invariants, the maximum and minimum values of $(d\mathbf{M}/ds)^2$ will fall where $\boldsymbol{\delta}$ has the greatest (positive or negative) projection on $\mathbf{h}$, that is at the point of tangency of planes tangent to the indicatrix and perpendicular to $\mathbf{h}$, and for this condition $\mathbf{h}\cdot\boldsymbol{\mu} = 0$. *The principal directions (definition 3) are therefore those for which* $d\mathbf{M}/ds$ *is a maximum or minimum in magnitude, as in the ordinary three dimensional case.* It may reasonably be asked whether such a condition as the maximum or minimum of $d\mathbf{M}/ds$ in magnitude is not more intimately connected with the surface than the similar conditions on the curvature of a normal section. Unfortunately the condition breaks down for the case $\mathbf{h} = 0$, but there are important theorems on principal directions in the three dimensional case which suggest that $\mathbf{h} = 0$ is a really exceptional case.[44]

It is not difficult to make a choice between the three generalizations

44 See, for example, Eisenhart, *Differential Geometry*, p. 143.

just mentioned. The first, which follows Kommerell, gives four directions on the surface which are not perpendicular and which in the case of three dimensions reduce to the principal directions and the asymptotic directions. We do not ordinarily associate common properties to these two sets of directions. The second definition, suggested above, also gives four directions and in this case the reduction is to the principal directions combined with their bisectors. We do not usually investigate these directions or associate common properties to them and the principal directions. One great advantage of the third generalization is that we have, as principal, two and only two directions at each point and these directions are perpendicular.

The differential equations of the principal directions as defined by $\mathbf{h}\cdot\boldsymbol{\mu} = 0$ are, from (87),

$$\Sigma_{rs}\mathbf{h}\cdot\mathbf{y}_{rs}\bar{\lambda}^{(r)}\lambda^{(s)} = 0.$$

By (63) and $\lambda_r = \Sigma a_{rs}\lambda^{(s)}$ we may write

$$\Sigma_{rst}\mathbf{h}\cdot\mathbf{y}_{rs}a_{r+1,\,t}\,\lambda^{(t)}\lambda^{(s)}(-1)^{(r+1)} = 0,$$

or

$$\Sigma_{rst}\mathbf{h}\cdot\mathbf{y}_{rs}(-1)^{r+1}a_{r+1,\,t}\,dx_t dx_s = 0, \tag{121}$$

Written out at length we have

$$\mathbf{h}\cdot[(\mathbf{y}_{11}a_{21} - \mathbf{y}_{21}a_{11})dx_1^2 + (\mathbf{y}_{11}a_{22} - \mathbf{y}_{22}a_{11})dx_1dx_2 \\ + (\mathbf{y}_{12}a_{22} - \mathbf{y}_{22}a_{12})dx_2^2] = 0.$$

This equation is similar to the ordinary equation except that $\mathbf{y}_{rs}$ replaces b_{rs} and the whole is multiplied by $\mathbf{h}$.

If the lines of curvature are taken as parametric lines,

$$\mathbf{h}\cdot\mathbf{y}_{11}a_{21} = \mathbf{h}\cdot\mathbf{y}_{21}a_{11}, \qquad \mathbf{h}\cdot\mathbf{y}_{12}a_{22} = \mathbf{h}\cdot\mathbf{y}_{22}a_{21}.$$

These equations, since $\mathbf{y}_{11}a_{22} - \mathbf{y}_{22}a_{11} \neq 0$, demand that $a_{12} = 0$ and $\mathbf{h}\cdot\mathbf{y}_{21} = 0$. *The condition that the lines of curvature be parametric is no longer* $a_{12} = 0$, $\mathbf{y}_{12} = 0$; *the normal* $\mathbf{y}_{12}$ *need merely be perpendicular to* $\mathbf{h}$. In all this work $\mathbf{h}$ may be replaced by its value $\Sigma_{rs}a^{(rs)}\mathbf{y}_{rs}$ if desired. Special considerations need to be developed for the case $\mathbf{h} = 0$.

45. Asymptotic lines. When we seek for a generalization for asymptotic lines we may consider the equation $\mathbf{h}\cdot\Psi = 0$, where Ψ is the second fundamental form, as defining asymptotic lines in general.

Indeed equation (78), namely, $d\mathbf{M}\cdot d\mathbf{M} = -Gds^2 + 2\mathbf{h}\cdot\Psi$ contains an important property of asymptotic lines on ordinary surfaces; the asymptotic lines are those for which the rate of turning of the normal, in this case the torsion, is $\sqrt{(-G)}$. Along the asymptotic directions in the general case of n dimensions, $d\mathbf{M}\cdot d\mathbf{M} = -Gds^2$, that is, the rate of turning of the tangent plane is $\sqrt{(-G)}$, if by analogy $(d\mathbf{M}/ds)^2$ may be called the rate of turning when successive planes do not intersect in a line.[45]

In the ordinary case $\boldsymbol{\alpha} = 0$ for the asymptotic lines. In the general case $\boldsymbol{\alpha}\cdot\mathbf{h} = 0$. To prove this consider [see (67)]

$$0 = \mathbf{h}\cdot\Psi = \mathbf{h}\cdot\Sigma_{rs}[\boldsymbol{\alpha}\lambda_r\lambda_s + \boldsymbol{\mu}(\lambda_r\bar{\lambda}_s + \bar{\lambda}_r\lambda_s) + \boldsymbol{\beta}\bar{\lambda}_r\bar{\lambda}_s]dx_r dx_s\,.$$

As $dx_r{:}dx_s = \lambda^{(r)}{:}\lambda^{(s)}$ for the curves defined by λ, these curves will be along the asymptotic directions when and only when $\mathbf{h}\cdot\boldsymbol{\alpha} = 0$, as the other terms vanish in the summation.

The condition $\mathbf{h}\cdot\boldsymbol{\alpha} = 0$ means that the curvature of the asymptotic line is perpendicular to $\mathbf{h}$ and consequently the osculating plane of the curve is perpendicular to $\mathbf{h}$. Conversely if the osculating plane is perpendicular to $\mathbf{h}$ then $\boldsymbol{\alpha}$ must be perpendicular to $\mathbf{h}$. Hence: *The asymptotic line is characterized by the property that its osculating plane is perpendicular to the mean curvature vector as in the three dimensional case.*

As in the case of principal directions (Definition 3), the asymptotic lines we have defined become illusory for minimal surfaces. *For surfaces, not minimal, the asymptotic lines cannot be orthogonal*, as may be seen from the configuration of the indicatrix.

That the condition $\mathbf{h}\cdot\boldsymbol{\alpha} = 0$ may be satisfied for a real direction on the surface, the plane π through the surface point O perpendicular to $\mathbf{h}$ must cut the indicatrix in real points. Now: *The asymptotic lines here defined for any surface are bisected by the principal directions* (Definition 3). For the plane π is parallel to $\boldsymbol{\mu}$ if $\mathbf{h}\cdot\boldsymbol{\mu} = 0$ and cuts the plane of the indicatrix in a line parallel to $\boldsymbol{\mu}$. The two vectors $\boldsymbol{\alpha}$ go to points of the indicatrix which represent equal amounts of the surface angle θ, above and below the directions for which $\mathbf{h}\cdot\boldsymbol{\mu} = 0$. If this plane π is tangent to the indicatrix the asymptotic lines fall together. *The condition that the asymptotic directions fall together (and*

45 If we define the angle between two plane vectors, whether or not these be simple planes, by the formula $\cos\theta = \mathbf{M}\cdot\mathbf{N}/(\mathbf{M}^2\mathbf{N}^2)^{\frac{1}{2}}$ we have a real angle whenever the planes or complexes are real. If $\mathbf{M}$ is a unit plane, $\mathbf{N}$ a nearby unit plane $\mathbf{M}+\Delta\mathbf{M}$, then $2\mathbf{M}\cdot\Delta\mathbf{M} + (\Delta\mathbf{M})^2 = 0$ and by a familiar transformation we find $(d\theta/ds)^2 = (d\mathbf{M}/ds)^2$.

coincide with one of the principal directions) is that $\mathbf{h}$ *shall be normal to a tangent plane to Cone* I, *hence an element of Cone* II, that is,

$$\mathbf{h}\cdot\Phi\cdot\mathbf{h} = 0 = (\mathbf{h}\cdot\mathbf{h})^2 - (\mathbf{h}\cdot\boldsymbol{\mu})^2 - (\mathbf{h}\cdot\boldsymbol{\delta})^2.$$

In ordinary surface theory this reduces to $h^2G = 0$; but here $\mathbf{h}$, $\boldsymbol{\mu}$, and $\boldsymbol{\delta}$ are not collinear and the condition is not satisfied by $G = 0$. Indeed,

$$\begin{aligned}\mathbf{h}\cdot\Phi\cdot\mathbf{h} - \mathbf{h}^2\Phi_S &= \mathbf{h}^2\boldsymbol{\mu}^2 - (\mathbf{h}\cdot\boldsymbol{\mu})^2 + \mathbf{h}^2\boldsymbol{\delta}^2 - (\mathbf{h}\cdot\boldsymbol{\delta})^2 \\ &= (\mathbf{h}\times\boldsymbol{\mu})^2 + (\mathbf{h}\times\boldsymbol{\delta})^2 = \mathbf{h}\cdot\Phi\cdot\mathbf{h} - \mathbf{h}^2Ga.\end{aligned}$$

We see therefore that surfaces for which $G = 0$ make $\mathbf{h}\cdot\Phi\cdot\mathbf{h}$ positive unless $\mathbf{h}$, $\boldsymbol{\mu}$, $\boldsymbol{\delta}$ are collinear, that is unless the surface is three dimensional at the point in question. The condition $\mathbf{h}\cdot\Phi\cdot\mathbf{h} > 0$ means, however, that the plane π perpendicular to $\mathbf{h}$ through O does not cut the indicatrix, that is that the asymptotic directions on the surface are imaginary. Hence: *For all developables except the twisted curve surfaces the asymptotic directions are imaginary.*

The special cases which arise when the surface is four dimensional, with the indicatrix either an ellipse or a linear segment are not especially different from the general case.

The scalar form $\mathbf{h}\cdot\Psi$ which for the definition of asymptotic directions (in our sense) has taken the place of the scalar form ψ (second fundamental form) in three dimensions may be used to define a conjugate system of curves upon the surface as in the ordinary three dimensional case. The asymptotic lines $\mathbf{h}\cdot\Psi = 0$ are then the double elements in the involution. It is easy to see that the lines of curvature (in our sense) are the pair of orthogonal elements in this involution. For if we use the lines of curvature (in our sense) as parametric curves the forms $\mathbf{h}\cdot\Psi$ and $\varphi = ds^2$ are simultaneously reduced to a sum of squares since $a_{12} = 0$ and $\mathbf{h}\cdot\mathbf{y}_{12} = 0$ in this case.

Kommerell's generalization of asymptotic directions in case $n = 4$ was to those directions which correspond to the infinite points of his characteristic (our Conic II, inverse pedal to the indicatrix), i. e., to directions which make the normal curvature $\boldsymbol{\alpha}$ tangent to the indicatrix, namely $\boldsymbol{\alpha}\times\boldsymbol{\mu} = 0$. On a surface in S_4 there are two such directions; but the generalization breaks down for $n > 4$ because the condition $\boldsymbol{\alpha}\times\boldsymbol{\mu} = 0$ cannot be satisfied.[46] We are therefore forced to

46 Kommerell's second fundamental form, the vanishing of which determines his asymptotic directions, has therefore no relation at all to *general* surface theory, because his asymptotic directions do not exist in general. The second fundamental forms which we develop are vital to the theory.

conclude that neither the principal directions nor the asymptotic lines as defined by Kommerell are the best generalizations of corresponding lines on ordinary surfaces. That we have found other and better definitions may be attributed in part to the broader view point that we get by working in higher than four dimensional space, but must be credited in large measure to the suggestiveness of the method of attack developed by Ricci in his *Lezioni*.

Kommerell's type of asymptotic lines will exist only when $\boldsymbol{\alpha}\times\boldsymbol{\mu} = 0$, that is, only when the indicatrix lies in a plane through the surface

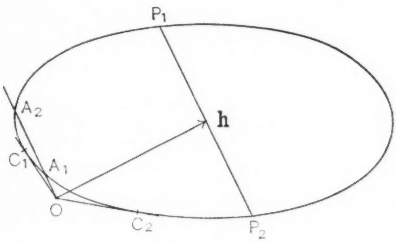

FIGURE 3.

point and the surface becomes four dimensional at the point. This condition has been discussed in §43. It will be seen that Kommerell's asymptotic lines are identical with Segre's characteristics; they are therefore important lines for those surfaces on which they exist. (Levi has asymptotic lines only in case the two characteristics coincide, their common direction being then called asymptotic.)

In the four dimensional case the important lines on the surface are as follows: Two principal directions corresponding to the points P_1 and P_2 for which $\mathbf{h}\cdot\boldsymbol{\mu} = 0$; two asymptotic directions (in our sense) A_1 and A_2 for which $\mathbf{h}\cdot\boldsymbol{\alpha} = 0$; two characteristic directions C_1 and C_2 for which $\boldsymbol{\alpha}\times\boldsymbol{\mu} = 0$. The principal directions are orthogonal and bisect the asymptotic directions, but need not bisect the characteristic directions; the asymptotic and characteristic directions divide each other harmonically. If O lies on the indicatrix, A_1, C_1, C_2 coincide. If the indicatrix reduces to a linear segment P_1 and C_1, P_2 and C_2 coincide.

46. The Dupin indicatrix. Another way of getting at the properties of a surface in ordinary space is by the Dupin indicatrix, which is the intersection of the surface by a tangent plane (or a plane parallel thereto). In five dimensions we must take a hyperplane (a four

dimensional linear spread) to cut the surface. If we consider a hyperplane $uz_1 + vz_2 + wz_3 = 0$ tangent to the surface in the standard form (109′); the intersection is,

$$u[h(x^2 + y^2) + e(x^2 - y^2)] + vf(x^2 - y^2) + w[A(x^2 - y^2) + 2Bxy] = 0.$$

There are ∞^2 such hyperplanes. The discriminant of the quadratic form

$$[u(h + e) + vf + wA]x^2 + [u(h - e) - vf - wA]y^2 + 2wBxy] = 0 \quad (127)$$

is

$$\Delta = w^2B^2 + (ue + vf + wA)^2 - u^2h^2.$$

The equation $\Delta = 0$ determines a quadric cone. Hence: *There are ∞^1 normal directions $u{:}v{:}w$ (forming a quadric cone) such that the tangent hyperplanes normal to any of these directions cut the surface in coincident directions.*

If $u{:}v{:}w$ be the directions of an element of Cone II we have, from (114),

$$(h^2 - e^2)u^2 - f^2v^2 - (A^2 + B^2)w^2 - 2fAvw - 2Aeuw - 2feuv = 0,$$

as the equation of Cone II (with its vertex transferred to O). This is identical with $\Delta = 0$ except for sign. We see therefore that: *The tangent hyperplanes which are perpendicular to the elements of Cone* II *cut the surface in coincident lines; these hyperplanes are also the tangent hyperplanes to Cone* I. Hence we may state: *The tangent hyperplanes which cut the indicatrix in real points cut the surface in real directions; those which cut the indicatrix in imaginary points cut the surface in imaginary directions; and those tangent to the indicatrix cut the surface in coincident directions.*

Particular interest attaches to the hyperplane ($z_1 = 0$) perpendicular to $\mathbf{h}$. This cuts the surface in the directions $(h + e)x^2 + (h - e)y^2 = 0$. These directions are real, coincident, or imaginary according as $h < e$, $h = e$, or $h > e$. This locus will be called the (generalized) *Dupin indicatrix.*

The condition $\boldsymbol{\alpha}\cdot\boldsymbol{\mu} = 0$ and $\boldsymbol{\mu}\cdot\boldsymbol{\delta} = 0$ which give the first two generalizations of principal directions may be calculated from (111), (112), but exhibit no special properties relative to the axes used in standardizing the equation of the surface. The condition $\mathbf{h}\cdot\boldsymbol{\mu} = 0$, however, is satisfied by the x and y axes in the tangent plane when the form

(109′) is used. We may say therefore that: *The principal directions (third generalization) at a point coincide with the principal directions of the three dimensional surface obtained by projecting the surface on the three space determined by the tangent plane and the mean curvature, or, are along the axes of the (degenerate) conic in which a tangent* S_{n-1} *normal to the mean curvature cuts the surface.* (The conic obtained as the intersection of any S_{n-1} which is normal to any line in the plane of $\mathbf{h}$ and the perpendicular $\Phi\cdot\mathbf{h}$ on the plane of $\boldsymbol{\mu}\times\boldsymbol{\delta}$ from O has the same axes.)

In the special case $h = 0$ the conic (122) always has $\Delta > 0$, and is an hyperbola. There is no real hyperplane which cuts the surface in a double direction. Cone II becomes a cylinder with elements perpendicular to the plane $\boldsymbol{\mu}\times\boldsymbol{\delta}$ of the indicatrix. From (109), $\boldsymbol{\mu}\times\boldsymbol{\delta} = B(e\mathbf{k}_3\times\mathbf{k}_1 - f\mathbf{k}_2\times\mathbf{k}_3)$. The hyperplanes perpendicular to any direction in the plane of the indicatrix have $w = 0$ and cut the surface in the same locus $x^2 - y^2 = 0$,— except the particular one for which $u:v = f:-e$ which causes (121) to vanish identically and contains all directions on the surface. We may therefore define, if we choose, the directions of the x and y axes as principal directions and the orthogonal directions $x^2 - y^2 = 0$ as asymptotic lines on the surface at the point where $h = 0$.

A reference to (111) shows that $\mathbf{h}\cdot\boldsymbol{\alpha} = 0$ means $h + e\cos 2\theta = 0$. On comparison with (122) we see that the directions θ for which $h + e\cos 2\theta = 0$ are the asymptotic directions of the intersection of the surface with the tangent S_{n-1} perpendicular to $\mathbf{h}$. Hence: *The asymptotic directions on a surface are the directions in which the surface is cut by a tangent hyperplane perpendicular to the mean curvature vector.* This gives added corroboration of the generalization of the Dupin indicatrix to the intersection of the surface and this particular S_{n-1}.

47. A second standard form for a surface. In the three dimensional theory the condition $G = 0$ is unchanged by the general linear transformation. This is no longer the case in higher dimensions. To discuss briefly projective properties of a surface we may proceed as follows. The general surface has the property that the tangent spaces S_{n-1} which cut the surface in a double direction envelope a nondegenerate cone. This statement is projective and the analytic statement is $\mathbf{h}\times\boldsymbol{\mu}\times\boldsymbol{\delta} \neq 0$. The condition $(\mathbf{h}\times\boldsymbol{\mu}\times\boldsymbol{\delta})^2 = 0$ is therefore invariant under projection. (The condition $\mathbf{h}\times\boldsymbol{\mu}\times\boldsymbol{\delta} = 0$ is the condition for the existence of Segre's characteristics, and as Segre was discussing projective properties, the result stated is but a corollary to

his work.) Now $(\mathbf{h}\times\boldsymbol{\mu}\times\boldsymbol{\delta})^2$ is Gibbs's invariant Φ_3 or $|\,\Phi\,|$ for the dyadic $\Phi = \mathbf{hh} - \boldsymbol{\mu\mu} - \boldsymbol{\delta\delta} = \frac{1}{2}\mathbf{y}_{11}\mathbf{y}_{22} + \frac{1}{2}\mathbf{y}_{22}\mathbf{y}_{11} - \mathbf{y}_{12}\mathbf{y}_{12}$. We may therefore write as the projective invariant

$$\Phi_3 = |\,\Phi\,| = \tfrac{1}{4}(\mathbf{y}_{11}\times\mathbf{y}_{22}\times\mathbf{y}_{12})^2 = 0.$$

In case $\Phi_3 \neq 0$, we can find a second standard form for the development of a surface about a point. It has been shown that if we project a surface on the S_3 determined by the tangent plane and a normal parallel to an element of Cone II, the projection has total curvature null. By taking an element of Cone III and two perpendicular elements of Cone II as axes, the expansion to second order terms becomes

$$\begin{aligned} z_1 &= \tfrac{1}{2}(Ax^2 + 2Bxy + Cy^2), & G &= AC - B^2, \\ z_2 &= \tfrac{1}{2}(A_1x^2 + 2B_1xy + C_1y^2), & 0 &= A_1C_1 - B_1^2, \\ z_3 &= \tfrac{1}{2}(A_2x^2 + 2B_2xy + C_2y^2), & 0 &= A_2C_2 - B_2^2. \end{aligned}$$

We shall show that by a proper choice of the element of Cone III, *the standard form*

$$z_1 = \tfrac{1}{2}(Ax^2 + 2Bxy + Cy^2), \quad z_2 = \tfrac{1}{2}Dx^2, \quad z_3 = \tfrac{1}{2}Ey^2 \tag{123}$$

may be found. All that is necessary to prove this is to prove that the two double lines obtained from $z_2 = 0$ and $z_3 = 0$ may be made perpendicular. If we set $\xi = uh$, $\eta = ue + vf + wA$, $\zeta = wB$. The condition $\Delta = 0$ becomes $\zeta^2 + \eta^2 - \xi^2 = 0$. We have to find two directions u, v, w, such that

$$\zeta_1^2 + \eta_1^2 - \xi_1^2 = 0, \qquad \zeta_2^2 + \eta_2^2 - \xi_2^2 = 0, \qquad u_1u_2 + v_1v_2 + w_1w_2 = 0.$$

Furthermore the double lines $(\xi + \eta)x^2 + (\xi - \eta)y^2 + 2\zeta xy = 0$ must be perpendicular for the two series of ξ, η, ζ. Hence, if ρ be a factor,

$$\rho(\xi_2 - \eta_2) = \xi_1 + \eta_1, \qquad \rho(\xi_2 + \eta_2) = \xi_1 - \eta_1, \qquad \rho\zeta_2 = -\zeta_1,$$

or $\quad \zeta_1\xi_2 - \zeta_1\eta_2 + \zeta_2\xi_1 + \zeta_2\eta_1 = 0, \qquad \zeta_1\xi_2 + \zeta_1\eta_2 + \zeta_2\xi_1 - \zeta_2\eta_1 = 0,$

or $\qquad \zeta_1\xi_2 + \zeta_2\xi_1 = 0, \qquad \zeta_1\eta_2 - \zeta_2\eta_1 = 0.$

Let $\xi_i/\zeta_i = \Xi_i$, $\eta_i/\zeta_i = H_i$; then the five equations are

$$H_1^2 - \Xi_1^2 + 1 = 0, \quad H_2^2 - \Xi_2^2 + 1 = 0, \quad \Xi_2 + \Xi_1 = 0, \quad H_2 - H_1 = 0$$

and

$$u_1u_2 + v_1v_2 + w_1w_2 = -\Xi_1^2\frac{f^2+c^2}{h^2} + H_1^2 - 2H_1\frac{A}{B} + \frac{f^2+A^2}{B^2} = 0.$$

These five equations are clearly consistent, $H_2^2 - \Xi_2^2 + 1 = 0$, being redundant. The actual solution could be carried out by finding H_1 first, then Ξ_1 and finally Ξ_2 and H_2. The solution is unique—the four apparently different solutions corresponding to changing the signs of u, v, w, and to interchanging the two sets. Hence (123) is established as a standard form.

The value of $\mathbf{h}$ is $2\mathbf{h} = (A + C)\mathbf{k}_1 + D\mathbf{k}_2 + E\mathbf{k}_3$, and of

$$\Phi = (AC - B^2)\mathbf{k}_1\mathbf{k}_1 + \tfrac{1}{2}CD(\mathbf{k}_1\mathbf{k}_2 + \mathbf{k}_2\mathbf{k}_1) + \tfrac{1}{2}AE(\mathbf{k}_1\mathbf{k}_3 + \mathbf{k}_3\mathbf{k}_1) + \tfrac{1}{2}DE(\mathbf{k}_2\mathbf{k}_3 + \mathbf{k}_3\mathbf{k}_2).$$

Here $\Phi_3 = \tfrac{1}{4}B^2E^2D^2$. If we carry out the linear transformation

$$x' = \alpha x, \quad y' = \beta y, \quad z_1' = \gamma z_1 + \delta z_2 + \epsilon z_3, \quad z_2' = \zeta z_2, \quad z_3' = \eta z_3, \tag{124}$$

the surface takes the form

$$z_1' = \frac{1}{2}\left(\frac{\gamma A + \delta D}{\alpha^2}x'^2 + \frac{2\gamma B}{\alpha\beta}x'y' + \frac{\gamma C + \epsilon E}{\beta^2}y'^2\right), \qquad z'_2 = \frac{1}{2}\frac{\zeta D}{\alpha^2}x'^2,$$
$$z'_3 = \frac{1}{2}\frac{\eta E}{\beta^2}y'^2.$$

The surface will be unaltered if the relations

$$\gamma = \alpha\beta, \quad \delta = A\alpha(\alpha - \beta)/D, \quad \epsilon = C\beta(\beta - \alpha)/E, \quad \zeta = \alpha^2, \quad \eta = \beta^2$$

are satisfied. There then ∞^2 transformations (124) which leave the surface unchanged in the neighborhood of the point O. Any of the ∞^2 transformations where

$$\gamma = \alpha\beta\frac{B'}{B}, \qquad \delta = \alpha\frac{A'\beta\alpha - B'A\beta}{DB}, \qquad \epsilon = \beta\frac{C'B\beta - B'C\alpha}{ED},$$
$$\zeta = \alpha^2\frac{D'}{D}, \qquad \eta = \beta^2\frac{E'}{E}$$

will carry the surface into one in which the five coefficients are any quantities A', B', C', D', E'. The determinant of the transformation

is $\Delta = \alpha^3\beta^3B'D'E'/BDE$, and hence the restriction on the quantities is merely that no one of α, β, B', D', E', shall vanish. We see that $\Phi'_3 \neq 0$ if $\Phi_3 \neq 0$; but that Φ_3 is not an invariant in the ordinary sense of projective geometry that $\Phi'_3 = \Delta^k\Phi_3$ — no more is G in the usual surface theory.

48. Surfaces of revolution. In higher dimensions the simplest type of rotation is that parallel to a plane, all the normals to the plane remaining fixed. If then $x = x(s)$, $z_i = z_i(s)$, $i = 1, 2, \ldots,$ be any twisted curve of which s is the arc, a surface of revolution

$$x = x(s)\cos\theta, \quad y = x(s)\sin\theta, \quad z_i = z_i(s)$$

may be obtained by the revolution of the curve parallel to the xy-plane. The surface is made up of circles parallel to the plane with radii equal to the distance of the twisted curve from the z-space of n-2 dimensions. The parameters of the surface are s and θ; the parametric curves are orthogonal. Further

$$d\mathbf{y} = (\mathbf{i}x'\cos\theta + \mathbf{j}y'\sin\theta + \Sigma\mathbf{k}_iz_i')ds + (-\mathbf{i}x\sin\theta + \mathbf{j}x\cos\theta)d\theta.$$

$$\mathbf{m} = \mathbf{i}x'\cos\theta + \mathbf{j}x'\sin\theta + \Sigma\mathbf{k}_iz_i', \qquad \mathbf{n} = -\mathbf{i}x\sin\theta + \mathbf{j}x\cos\theta,$$

$$\mathbf{p} = \mathbf{i}x''\cos\theta + \mathbf{j}x''\sin\theta + \Sigma\mathbf{k}_iz_i'', \quad \mathbf{q} = -\mathbf{i}x'\sin\theta + \mathbf{j}x'\cos\theta,$$

$$\mathbf{r} = -\mathbf{i}x\cos\theta - \mathbf{j}x\sin\theta.$$

As $x'^2 + \Sigma z'^2 = 1$, we have $x'x'' + \Sigma z'z'' = 0$, and

$$\mathbf{m}^2 = 1, \quad \mathbf{m}\cdot\mathbf{n} = 0, \quad \mathbf{n}^2 = x^2, \quad \mathbf{m}\cdot\mathbf{p} = 0, \quad \mathbf{m}\cdot\mathbf{q} = 0,$$

$$\mathbf{m}\cdot\mathbf{r} = -xx', \quad \mathbf{n}\cdot\mathbf{p} = 0, \quad \mathbf{n}\cdot\mathbf{q} = xx', \quad \mathbf{n}\cdot\mathbf{r} = 0,$$

$$a_{11} = 1, \quad a_{12} = 0, \quad a_{22} = x^2, \quad a = x^2,$$

$$\mathbf{y}_{11} = \mathbf{p}, \quad \mathbf{y}_{12} = \mathbf{q} - x'\mathbf{m}/x = 0, \quad \mathbf{y}_{22} = \mathbf{r} + xx'\mathbf{m}.$$

The element of arc is $ds^2 + x^2d\theta^2$. It therefore appears that: *The surface of revolution is always applicable upon a surface of revolution in three dimensions* in which the directrix in the xy-plane is $x = x(s)$, $z = z(s)$. [The equation $z = z(s)$ is redundant and so is one of the $n - 2$ equations $z_i = z_i(s)$].

The value of G is $\mathbf{q}\cdot\mathbf{r}/a = -x''/x$. The condition $G = 0$ for a developable is therefore $x'' = 0$ or $x = c_1s + c_2$ which establishes between the differentials the relation $dx = c_1ds$ or

$$(1 - c_1^2)dx^2 = c_1^2(dz_1^2 + dz_2^2 + \ldots + dz_{n-2}^2), \qquad c_1 < 1.$$

In case $n = 3$ the solution is immediate, viz. $z = mx + b$, a line. In case $n > 3$ we may assign to $n - 3$ of the variables z_i any arbitrary values as functions of x (provided that the sum of dz^2 is not too large if we desire a real surface). For instance if we consider the case $n = 4$ and let $z_1 = a_1 \cos bx$,

$$(1 - c_1^2 - c_1^2 a_1^2 b^2 \sin^2 bx) dx^2 = c_1^2 dz_2^2, \quad \text{or} \quad (1 - c_1^2) \cos^2 bx dx^2 = c_1^2 dz_2^2$$

if we choose $c_1^2 a_1^2 b^2 = (1 - c_1^2)$ to simplify the integration for a particular case. Then

$$z_2 = \frac{\sqrt{1 - c_1^2}}{c_1 b} \sin bx + C = a_1 \sin bx + C.$$

The curve $z_1 = a_1 \cos bx$, $z_2 = a_1 \sin bx$ is a circular helix about the axis of x in the xz_1z_2 space. The four dimensional surface of revolution is

$$z_1 = a_1 \cos b \sqrt{x^2 + y^2}, \qquad z_2 = a_1 \sin b \sqrt{x^2 + y^2}.$$

We see therefore that: *The developables of revolution when $n > 3$ form an extended class of surfaces instead of reducing merely to the cones and cylinders.*

The value of **h** is given by

$$2\mathbf{h} = \mathbf{p} + (\mathbf{r} + xx'\mathbf{m})/x^2.$$

If we designate $\mathbf{i} \cos\theta + \mathbf{j} \sin\theta$ by $\mathbf{u}$, a unit vector,

$$\mathbf{p} = x''\mathbf{u} + \Sigma \mathrm{k}_i z_i'', \quad \mathbf{r} = -x\mathbf{u}, \quad \mathbf{m} = x'\mathbf{u} + \Sigma \mathbf{k}_i z_i'$$

and

$$2\mathbf{h} = \mathbf{u}(xx'' + x'^2)/x + \Sigma \mathbf{k}_i (z_i'' + x'z'/x).$$

The condition $\mathbf{h} = 0$ for a minimal surface therefore is

$$xx'' - 1 + x'^2 = 0, \quad z_i'' + x'z_i'/x = 0.$$

The last equation shows that $z_i'x = c_i$, the first that $x^2 = (s + b)^2 + a^2$. Hence

$$\frac{z_i + K_i}{c_i} = \cosh^{-1}\frac{x}{a}, \quad i = 1, 2, \ldots, n - 2,$$

with the condition $\Sigma c_i^2 = a^2$. From this it follows that

$$\frac{z_1 + K_1}{c_1} = \frac{z_2 + K_2}{c_2} = \ldots = \frac{z_{n-2} + K_{n-2}}{c_{n-2}}.$$

The curve therefore lies in a plane through the x axis and some line in the z space; it is the common catenary and the result is: *The only minimal surface of revolution is the ordinary catenoid.*[47]

As $a_{12} = 0$ and $\mathbf{y}_{12} = 0$, $\boldsymbol{\mu} \times \boldsymbol{\delta} = 0$. *All surfaces of revolution are of the type for which the indicatrix reduces to a linear segment.* Our lines of curvature coincide with Segre's characteristics and both lie along the circles and the various directions assumed by the directrix in the revolution.

49. Note on a vectorial method of treating surfaces. Another general method of dealing with the theory of surfaces upon a vector basis may be mentioned without going much into details. In the ordinary three dimensional case we set up the linear vector function Φ which expresses the differential normal $d\mathbf{n}$ in terms of the displacement $d\mathbf{r}$, i. e., $d\mathbf{n} = d\mathbf{r}\cdot\Phi$. As the properties of dyadics Φ are well known many properties of the surface may be found at once.[48]

In the general case the tangent plane $d\mathbf{M}$ is connected linearly with the displacement $d\mathbf{r}$. In fact if $d\mathbf{r} = \mathbf{m}\, du + \mathbf{n}\, dv$, a differentiating operator

$$\nabla = \mathbf{n}\cdot\mathbf{M}\frac{\partial}{\partial u} - \mathbf{m}\cdot\mathbf{M}\frac{\partial}{\partial v} \tag{125}$$

may be written down which is invariant under a change of parameters. (This is obvious since

$$d\mathbf{r}\cdot\nabla = \mathbf{m}\cdot(\mathbf{n}\cdot\mathbf{M})du\frac{\partial}{\partial u} - \mathbf{n}\cdot(\mathbf{m}\cdot\mathbf{M})dv\frac{\partial}{\partial v} = du\frac{\partial}{\partial u} + dv\frac{\partial}{\partial v},$$

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47 A geometric proof may be given as follows. Since the surface is minimum $\mathbf{h} = 0$, and since the surface is of revolution the indicatrix reduces to a segment (see below). Consequently the minimum surface of revolution is one for which every point is axial with the center of the indicatrix (not its end) at the surface point. Hence the minimum surface of revolution must lie in three dimensions and in this case the surface is known to be a catenoid.

48 For a brief discussion see Gibbs-Wilson, *Vector Analysis*, p. 411 ff.

which makes $d\mathbf{r}\cdot\nabla = d$. If desired it is possible to remove the condition that **M** be a unit tangent plane by writing

$$\nabla = \frac{\mathbf{n}\cdot\mathbf{M}}{\mathbf{M}^2}\frac{\partial}{\partial u} - \frac{\mathbf{m}\cdot\mathbf{M}}{\mathbf{M}^2}\frac{\partial}{\partial v},$$

where **M** is $\mathbf{m}\times\mathbf{n}$) We have then, in the general case where $n > 3$,

$$d\mathbf{M} = d\mathbf{r}\cdot\nabla\mathbf{M} = d\mathbf{r}\cdot\Lambda, \quad \Lambda = \nabla\mathbf{M}, \tag{126}$$

where **M** is the unit tangent plane, to correspond to $d\mathbf{n} = d\mathbf{r}\cdot\Phi$ in the particular case $n = 3$.

The dyadic Λ, however, is one in which the antecedent vectors in the dyads are 1-vectors and the consequent vectors are 2-vectors, i. e., planes, simple or otherwise,—

$$\Lambda = \mathbf{n}\cdot\mathbf{M}\frac{\partial\mathbf{M}}{\partial u} - \mathbf{m}\cdot\mathbf{M}\frac{\partial\mathbf{M}}{\partial v}\cdot \tag{127}$$

Further

$$\frac{d\mathbf{M}}{ds} = \frac{d\mathbf{r}}{ds}\cdot\Lambda = \mathbf{t}\cdot\Lambda,$$

where **t** is a unit tangent 1-vector in any direction. The rate of change of the tangent plane in the direction **t** is therefore $\mathbf{t}\cdot\Lambda$.

The properties of a 1–2 dyadic such as Λ are not well known and the development of the surface theory from this point of view is therefore hampered. Some points, however, are readily ascertained. First, there is an invariant or covariant line (1-vector) and an invariant space (3-vector) obtained from Λ by the familiar processes of inserting the signs of scalar and vector products between the elements of the dyadic,— thus

$$\begin{aligned} \mathbf{l} &= (\mathbf{n}\cdot\mathbf{M})\cdot\frac{\partial\mathbf{M}}{\partial u} - (\mathbf{m}\cdot\mathbf{M})\cdot\frac{\partial\mathbf{M}}{\partial v} \\ \mathbf{S}_3 &= (\mathbf{n}\cdot\mathbf{M})\times\frac{\partial\mathbf{M}}{\partial u} - (\mathbf{m}\cdot\mathbf{M})\times\frac{\partial\mathbf{M}}{\partial v} \end{aligned} \tag{128}$$

By the transformation $(\mathbf{b}\cdot\mathbf{C})\cdot\mathbf{A} = -(\mathbf{C}\cdot\mathbf{A})\mathbf{b} + \mathbf{C}\cdot(\mathbf{b}\times\mathbf{A})$,

$$\mathbf{l} = -\mathbf{n}\mathbf{M}\cdot\frac{\partial\mathbf{M}}{\partial u} + \mathbf{M}\cdot\left((\mathbf{n}\times\frac{\partial\mathbf{M}}{\partial u}\right) + \mathbf{m}\mathbf{M}\cdot\frac{\partial\mathbf{M}}{\partial v} - \mathbf{M}\cdot\left(\mathbf{m}\times\frac{\partial\mathbf{M}}{\partial v}\right)\cdot$$

As $\mathbf{M}^2 = 1$, the first and third terms vanish; and as $\partial\mathbf{M}/\partial u$ and $\partial\mathbf{M}/\partial v$ are perpendicular to $\mathbf{M}$, so must the spaces $\mathbf{n}\times\partial\mathbf{M}/\partial u$ and $\mathbf{m}\times\partial\mathbf{M}/\partial v$ be perpendicular to $\mathbf{M}$, and the other two terms will vanish. Hence: *The vector invariant* $\mathbf{l}$ *of the dyadic* Λ *vanishes.*

The invariant 3-vector $\mathbf{S}_3$ may be calculated. The work may be simplified by taking the parameter curves orthogonal with u and v equal to the arc along these curves (except for infinitesimals) in the neighborhood of any preassigned point.

Then $\mathbf{m} = \boldsymbol{\xi}$, $\mathbf{n} = \boldsymbol{\eta}$ and

$$\begin{aligned}\mathbf{S}_3 &= [\mathbf{n}\cdot(\boldsymbol{\xi}\times\boldsymbol{\eta})] \times \frac{d(\boldsymbol{\xi}\times\boldsymbol{\eta})}{ds} - [\boldsymbol{\xi}\cdot(\boldsymbol{\xi}\times\boldsymbol{\eta})] \times \frac{d(\boldsymbol{\xi}\times\boldsymbol{\eta})}{ds} \\ &= [\mathbf{n}\cdot(\boldsymbol{\xi}\times\boldsymbol{\eta})] \times [\boldsymbol{\alpha}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\mu}] - [\boldsymbol{\xi}\cdot(\boldsymbol{\xi}\times\boldsymbol{\eta})] \times [\boldsymbol{\mu}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\beta}] \\ &= \boldsymbol{\xi}\times\boldsymbol{\alpha}\times\boldsymbol{\eta} + \boldsymbol{\eta}\times\boldsymbol{\xi}\times\boldsymbol{\beta} = -(\boldsymbol{\xi}\times\boldsymbol{\eta}) \times (2\mathbf{h}) = -2\mathbf{M}\times\mathbf{h}.\end{aligned}$$

Hence: *The invariant 3-vector* $\mathbf{S}_3 = \Lambda_\times$ *is* $-2\,\mathbf{M}\times\mathbf{h}$, *the space of the tangent plane and the mean curvature, and of magnitude equal to the mean curvature.*

Other invariants of the dyadic

$$\Lambda = \boldsymbol{\xi}(\boldsymbol{\alpha}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\mu}) + \boldsymbol{\eta}(\boldsymbol{\mu}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\beta})$$

are the dyadics $\Lambda\cdot\Lambda_C$, $\Lambda_C\cdot\Lambda$ $\Lambda{:}\Lambda_C$, and so on, and the quantities obtained from them by inserting dots and crosses. For instance,

$$\Lambda_C\cdot\Lambda = (\boldsymbol{\alpha}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\mu})\,(\boldsymbol{\alpha}\times\boldsymbol{\xi} + \boldsymbol{\xi}\times\boldsymbol{\mu}) + (\boldsymbol{\mu}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\beta})\,(\boldsymbol{\mu}\times\boldsymbol{\eta} + \boldsymbol{\xi}\times\boldsymbol{\beta}),$$

and $$\mathbf{T}_4 = (\Lambda_C\cdot\Lambda)_\times = -2\mathbf{M}\times[(\boldsymbol{\alpha} - \boldsymbol{\beta})\times\boldsymbol{\mu}] = 4\mathbf{M}\times\boldsymbol{\mu}\times\boldsymbol{\delta},$$

$$\Lambda{:}\Lambda_C = 2(\boldsymbol{\delta}\boldsymbol{\delta} + \boldsymbol{\mu}\boldsymbol{\mu} - \mathbf{h}\mathbf{h}).$$

Hence $-\mathbf{M}\cdot\mathbf{S}_3 = 2\mathbf{h}$, $\mathbf{M}\cdot\mathbf{T}_4 = 4\boldsymbol{\mu}\times\boldsymbol{\delta}$, $-\Lambda{:}\Lambda_c = 2\Phi/a$ are the quantities, found directly from the fundamental dyadic Λ, which have been found of prime importance in the theory of surfaces.[49]

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49 The line of development here followed is the inverse of that which would be followed in developing the surface theory from Λ. It is for brevity that we choose merely to verify that the already known quantities $\mathbf{h}$, $\boldsymbol{\mu}\times\boldsymbol{\delta}$, and Φ of surface theory may be derived from Λ.

One of the first problems in discussing 1–2 dyadics of the type $\mathbf{aA} + \mathbf{bB}$ would be the establishment of a standard form. If $\mathbf{a}$ and $\mathbf{b}$ be replaced by linear combinations $x\mathbf{a}' + y\mathbf{b}'$, $x'\mathbf{a}' + y'\mathbf{b}'$ of two vectors in their plane, the dyadic becomes

$$\Lambda = \mathbf{a}'(x\mathbf{A} + x'\mathbf{B}) + \mathbf{b}'(y\mathbf{A} + y'\mathbf{B}) = \mathbf{a}'\mathbf{A}' + \mathbf{b}'\mathbf{B}',$$

where $\mathbf{A}'$, $\mathbf{B}'$ are linear combinations of $\mathbf{A}$ and $\mathbf{B}$. We may then consider that for the antecedents $\mathbf{a}'$, $\mathbf{b}'$ we have chosen unit normal vectors $\mathbf{i}$, $\mathbf{j}$ so that $\Lambda = \mathbf{iA} + \mathbf{jB}$. If a rotation is carried out on $\mathbf{i}$, $\mathbf{j}$, we have

$$\Lambda = \mathbf{i}'(\mathbf{A}\cos\varphi - \mathbf{B}\sin\varphi) + \mathbf{j}'(\mathbf{A}\sin\varphi + \mathbf{B}\cos\varphi) = \mathbf{i}'\mathbf{A}' + \mathbf{j}'\mathbf{B}'.$$

The condition $\mathbf{A}' \cdot \mathbf{B}' = 0$, i. e., the condition that the consequents be orthogonal is that φ be determined from

$$\tan 2\varphi = 2\mathbf{A}\cdot\mathbf{B}/(\mathbf{A}^2 - \mathbf{B}^2),$$

which gives four values of φ spaced at right angles. Hence: *We may reduce* Λ *to the form*

$$\Lambda = \mathbf{iA} + \mathbf{jB}, \quad \mathbf{A}\cdot\mathbf{B} = 0, \tag{129}$$

and this reduction is unique (except for the indeterminateness of an interchange of $\mathbf{i}$ and $\mathbf{j}$ or a reversal of the sign of either). The reduction is wholly indeterminate when φ is indeterminate, i. e., when $\mathbf{A}\cdot\mathbf{B} = 0$ and $\mathbf{A}^2 = \mathbf{B}^2$. *In the special reduced form* (129), *the directions* $\mathbf{i}$, $\mathbf{j}$ *are along the principal directions on the surface in case we use for principal directions the definition 3 introduced and preferred by us.*[50]

If $\mathbf{j}$ is chosen relative to $\mathbf{i}$ and $\mathbf{M}$ so that $\mathbf{M} = \mathbf{i}\times\mathbf{j}$, we have

$$\boldsymbol{\alpha} = \mathbf{i}\cdot\Lambda\cdot\mathbf{j}, \quad \boldsymbol{\beta} = -\,\mathbf{j}\cdot\Lambda\cdot\mathbf{i}, \quad \boldsymbol{\mu} = -\,\mathbf{i}\cdot\Lambda\cdot\mathbf{i} = \mathbf{j}\cdot\Lambda\cdot\mathbf{j}.$$

As $\mathbf{ij} - \mathbf{ji}$ is a dyadic independent of the directions of $\mathbf{i}$ in $\mathbf{M}$,

$$2\mathbf{h} = (\mathbf{ij} - \mathbf{ji}):\Lambda = \boldsymbol{\alpha} + \boldsymbol{\beta}$$

50 Note the correspondence with three dimensions. If we have the dyadic $\mathbf{a}\boldsymbol{\alpha} + \mathbf{b}\boldsymbol{\beta} = \Phi$ where $d\mathbf{n} = d\mathbf{r}\cdot\Phi$, we may reduce to the form $\mathbf{i}\boldsymbol{\alpha} + \mathbf{j}\boldsymbol{\beta}$, $\boldsymbol{\alpha}\cdot\boldsymbol{\beta} = 0$, which is a reduction to the principal axes of Φ, and have then $\mathbf{i}$, $\mathbf{j}$ along the principal directions, since Φ is self-conjugate.

is an invariant of Λ, as verified above in a different way. Further $2\delta = \mathbf{i}\cdot\Lambda\cdot\mathbf{j} + \mathbf{j}\cdot\Lambda\cdot\mathbf{i}$ may by a rotation of $\mathbf{i}, \mathbf{j}$ into $\mathbf{i}', \mathbf{j}'$ be seen to describe a conic, our indicatrix.

These brief remarks must suffice to indicate the ease and directness of the vector method of discussing surfaces through the use of the dyadic $\Lambda = \nabla\mathbf{M}$ which is determined by the relation $d\mathbf{M} = d\mathbf{r}\cdot\Lambda$, where ∇ is a sort of surface differentiation built from analogy with the ordinary ∇ and defined identically with it by the equation $d = d\mathbf{r}\cdot\nabla$.

Massachusetts Institute of Technology,
Boston, Mass., *March*, 1916.